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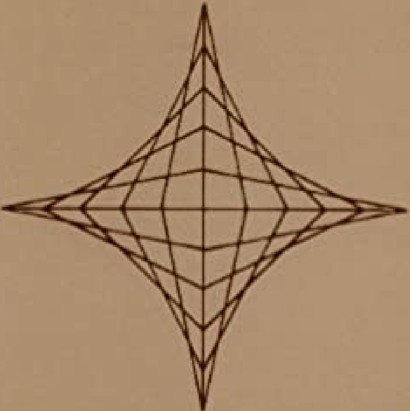
THE LOGIC of INCONSISTENCY

Nicholas Rescher
& Robert Brandom



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THE LOGIC OF INCONSISTENCY

A Study in Non-Standard Possible-World Semantics and Ontology

NICHOLAS RESCHER
AND
ROBERT BRANDOM

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For
GEORG HENRIK VON WRIGHT

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PREFACE

The evolution of the book was as follows. In the spring of 1977, Rescher gave to Brandom for reaction and criticism an essay amounting to a draft version of sections 1–12 and 26 of the book, which he had written during the preceding year. Brandom responded with various ideas for extension and application of this machinery, and Rescher suggested the metamorphosis of his essay into a joint book. On the basis of further exchanges and discussions, Brandom then proceeded to draft sections 13–25 and the Appendices during the summer of 1977. Both authors then revised their respective contributions and sought to blend the material into a continuous whole. Professors Gerald Massey, Carl Posy, Graham Priest, and Teddy Seidenfeld were particularly helpful and generous with their time during this period, and are responsible for a number of improvements in the final text, as is an anonymous publisher's reader. In the event, however, the responsibility for the materials of sections 1–12 remains with Rescher, while that for sections 13–25 and the Appendices belongs to Brandom. Section 26 was revised by Brandom on the basis of a draft by Rescher. (A translation of an earlier version of Rescher's contribution was published in an Italian anthology: Diego Marconi [ed.], *La Formalizzazione della Dialettica* [Torino: Rosenberg and Sellier, 1979].)

The origins of Rescher's contributions to this discussion lie in his book on *Cognitive Systematization* (Oxford: Basil Blackwell, 1979). When working on that project, it struck him that all of the parameters of systematicity—completeness, comprehensiveness, coherence, unity, simplicity, regularity, etc.—were matters of degree, with one seeming exception, that of consistency. And it seemed to him important to investigate whether consistency is indeed *sui generis* in setting up an absolute and altogether non-negotiable requirement of rational systematization.

Pittsburgh, PA
October 1978

PREFACE

The division of labor [illegible] as follows. In the spring of 19[illegible] Kleene gave [illegible] lectures [illegible] [illegible] a [illegible] of the [illegible] and [illegible] with [illegible] [illegible] and Kleene suggested the [illegible] [illegible]. On the basis of further exchanges and discussions [illegible] then proceeded to draft sections [illegible] and the [illegible] during the summer of 19[illegible]. Both authors then [illegible] their respective contributions [illegible] to mold the material into a [illegible] whole. Professors [illegible], Carl [illegible] and [illegible] were particularly helpful and generous with their time during the period [illegible] a number of improvements [illegible] the text [illegible] [illegible] responsibility for the [illegible] of [illegible] Kleene [illegible] and the [illegible] Section [illegible] [illegible] of an earlier version of [illegible] was published in [illegible] anthology [illegible] (Amsterdam [illegible] 19[illegible]).

The [illegible] of [illegible] contribution [illegible] book on [illegible] [illegible] [illegible] 19[illegible]. [illegible] of [illegible] [illegible] [illegible] [illegible], [illegible] of degree, with the [illegible] [illegible] of [illegible] [illegible] to [illegible] [illegible] and [illegible] [illegible].

[illegible]

[illegible]

COGITATIONS

"Do I contradict myself?
Very well, then, I contradict myself.
(I am large, I contain multitudes.)"
Walt Whitman

"Logic teaches us to expect some residue of dreaminess in the world, and even selfcontradictions."

C.S. Peirce, *Collected Papers*, Vol. IV, sect. 4.79.

"[M]any of the apparent contradictions and conflicts in Peirce's work result not from any feebleness of intellect or memory on his part, nor in most cases from any basic weakness in his philosophy, but rather from his almost incredible and many-sidedness, and a genuinely felt sympathy with widely varying viewpoints."

William H. Davis, *Peirce's Epistemology* (The Hague, 1972), p. 50.

"I seldom care to be consistent. In a philosophical view, consistency is a certain level at all times, maintained in all thoughts of one's mind. But, since nature is nearly all hill and dale, how can one keep naturally advancing in knowledge without submitting to the natural inequalities in the progress? Advance into knowledge is just like advance upon the grand Erie Canal, where, from the character of the country, change of level is inevitable; you are locked up and locked down with perpetual inconsistencies, and yet all the time you get on; while the dullest part of the whole route is what the boatmen call the 'long level'—a consistently-flat surface of sixty miles through stagnant swamps."

Melville's parody of Emerson in *The Confidence-Man*, ch. 36.

"It was to Hegel's credit that in his logic he tried to express the restless stream of events. But he was mistaken when he thought that this could not be reconciled with the principle of contradiction; insoluble contradictions only arise if one wants *to explain* the fact that life flows."

Wilhelm Dilthey: Selected Writings, Tr. H.P. Richman (Cambridge, 1970), p. 200.

"In the discussion of a text such as that of the *Agamemnon* the antagonism of two, and sometimes more, contrasting views is not due, as the inexperienced might think, to the *obscura diligentia* of classical scholars, but is the true expression of the complexity of the object itself. For a moment it might be possible by ingenious dialectical manoevers to concentrate the light on one possible interpretation of a passage and darken the opposite one, but the apparently defeated side will in the course of time come to life again and take its revenge on those who have neglected it."

Edward Frankel, Preface to
Aeschylus: Agamemnon
(3 vols., Oxford, 1950).

"Indeed, even at this stage, I predict a time when there will be mathematical investigations of calculi containing contradictions, and people will actually be proud of having emancipated themselves from consistency."

Ludwig Wittgenstein (1930), p. 332.

GLOSSARY OF NOTATION

$\sim$, &, $\vee$, $\supset$, $\equiv$ the familiar connectives of classical propositional logic

$\neg$, $\wedge$, $\vee$, $\Rightarrow$ meta-level sentential connectives [See p. 15.]

$\vdash$ assertability (in classical logic)

$=$, $\neq$ identity and nonidentity

p, *q*, *r*, ... proportional variables

P, *Q*, *P′*, *Q′*, ... meta-level sentential variables

R, *R′*, ... relations

x, *y*, *z*, ... individual variables

F, *G*, *H*, ... property variables

w_1, w_2, w_3, ... possible worlds

w^* the real or actual world

$\forall$, $\exists$ universal and existential quantifiers

$\forall^w$, $\exists^w$ world-relative quantifiers over individuals [See p. 32.]

$\cap$, $\cup$, $\in$, $\notin$ the familiar connectives of set theory

$\dot{\cap}$, $\dot{\cup}$ world-schematization and world-superposition [See pp. 9–10.]

{*X*/ ... *X* ...} the set of all *X* meeting the condition ... *X* ...

λ predicational abstraction (λxFx is the property of all *x* that have *F*, i.e. *F*-ness) [See p. 39.]

$[P]_w$ the obtaining status of *P* in the world *w* (either + for "obtains" or − for "does not obtain") [See p. 6.]

$\{P\}_w$ the ordered pair $[P]_w$, $[\sim P]_w$ [See p. 6.]

$[\![P]\!]$ (relative to w_1, w_2) the ordered pair $[P]_{w1}$, $[P]_{w2}$ [See p. 11.]

$\langle X_1, X_2, ..., X_4\rangle$ the ordered n-tuple: X_1, X_2, ..., X_4

$t_w(P)$ *P* is true in *w* [See p. 15.]

$/P/_w$ the truth-value of *P* in *w*; it may be *T* for "true" or *F* for "false" or possibly undefined [See pp. 27–28.]

I, $\not I$ model-structural identity or nonidentity [See p. 81.]

SECTION 1
Introduction

Since Aristotle's day, virtually all logicians and logically concerned philosophers in the mainstream of the Western tradition have had a phobia of inconsistency. They have been near to unanimous in proscribing it from the precincts of their logical and ontological theorizing, holding that the toleration of inconsistencies would inevitably bring cognitive disaster in its wake. The present discussion is intended as a contribution to the small but growing body of current thought which holds that this is by no means the case. Its main aim is to show that we can live with the prospect of inconsistency—not only in epistemology, but even in ontology.

Two great warring traditions regarding consistency originated in the days of the Presocratics at the very dawn of philosophy. The one, going back to Heraclitus, insists that the world *is not* a consistent system and that, accordingly, coherent knowledge of it cannot be attained by man. The doctrines of Heraclitus launched a tradition which runs through the Greek sceptics and their Renaissance successors to Hegel and especially those neo-Hegelians (especially the Marxists) who hold with Engels that there are real contradictions in nature.

The second tradition, going back to Parmenides, holds that the world *is* a consistent system and that knowledge of it must correspondingly be coherent as well, so that all contradictions must be eschewed. Aristotle here stands at the head of a great host, embracing not only his own school but the Stoics and virtually all later logicians outside the specifically Hegelian orbit who insist that this world of ours is self-consistent, and that the domain of reality is wholly free from inconsistency and self-contradiction. This thesis that absence of internal contradiction is a key test of objective reality can be found in Parmenides and Plato, in various sectors of the skeptical tradition, in rationalists like Leibniz, in empiricists like Locke, in idealists like Bradley, and among the great bulk of our own contemporaries of all persuasions. Let Bradley speak for them all:

> Ultimate reality is such that it does not contradict itself; here is an absolute criterion. And it is proved absolute by the fact that, either in endeavouring to deny it, or even in attempting to doubt it, we tacitly assume its validity. (*Appearance and Reality* [Oxford: The Clarendon Press, 1893], p. 120.)

What, after all, would an inconsistent world be like—one that violates the requisite of self-consistency in the strong (logical, Aristotelian) sense? Such a world would have to have the feature that in some perfectly definite way something both is and is not so. Of course, a substance can be hard by one standard of comparison and not hard by another; but, granted a definite and unequivocal specification of the respect at issue, then—so we are told—only one outcome in point of a characterization as hard or not hard is possible. (None of your subterfuges about something being both malleable and not malleable because malleable in one environment and not malleable in another.) The focal idea and leading principle here is the Aristotelian "Law of Contradiction"—or, better, of *Non-Contradiction*—is that something cannot both be and not be so unequivocally and in one and the selfsame respect.[1]

We are usually told that this position is inescapable because the consequences of its rejection are too horrendous to contemplate. But let us contemplate them anyhow. Let us assume that the worst has come to the worst and that we are confronted with the situation where there is an outright logical conflict in the characterization of an existential situation. Suppose, in short, that *reality* is inconsistent—that once all the relevant issues about sameness of respect are settled something both is and is not so in the ontological make-up of the world. What then?

The invocation of *ontology* here is significant. It indicates that our concern is with hard, *existential* inconsistency, and not with soft, *epistemological* inconsistency. Thus if $t(P)$ were to be construed not as "P is true in itself," but as "X (I, you, he, etc.) *maintains* that P is true" or again "P is true for all that X (I, you, he, etc.) maintains," then we can clearly and unproblematically have both $t(P)$ and $t(\sim P)$; these two seemingly conflicting contentions are now quite compatible. Traditional philosophic wisdom has it that while conflicts can indeed arise on such *epistemic* readings, this cannot happen *ontologically*.[2] It could not happen in one and the same world w that $t(P)$ and $t(\sim P)$ both obtain in the strict, "really-and-truly" sense (at the same time, same context, etc.). Is it not clear—indeed, is it not a matter of logical necessity—that this traditional wisdom is correct?

The present discussion will argue that such an outright rejection of ontological inconsistency is certainly not necessary—and perhaps not even desirable in the systematic scheme of things.[3] We shall maintain that it just is not an indispensable requisite of rationality to follow those—from Plato to Bradley—who consign whatever has an admixture of inconsistency to the realm of mere Appearance and rigidly exclude it from the sphere of Reality.

SECTION 2

Two Modes of Non-Standardness

Two classical principles regarding possible worlds stand at the forefront of traditional ontology:

(I) *Law of Excluded Middle*
Given a world w, then for any proposition P, either P obtains in w or its negation ($\sim P$) obtains in this world. [That is, of the pair P, $\sim P$ *at least one* obtains in w—there is no other possibility: *tertium non datur*.] (Note that to say that a proposition obtains in a world w is short for saying that the state of affairs purported by that proposition obtains in it.)

(II) *Law of Contradiction*
Given a world w, then for any proposition P, either P or $\sim P$ fails to obtain in w. [That is, of the pair P, $\sim P$ *at most one* obtains in w.]

Jointly these two principles produce the circumstance that, given a world w, then for any proposition P, exactly one member of the pair P, $\sim P$ obtains in w.

We shall characterize those worlds for which both of these principles hold as the *standard* worlds. Such standard worlds are *status-decisive* for propositions: Given any proposition P, either it or its negation $\sim P$ will obtain, and the other will fail, so that the obtaining-status of P is unequivocally determined. Such standard worlds represent the orthodox case envisaged almost invariably by ontologists from Aristotle to McTaggart and beyond.

In calling this tradition into doubt, the present discussion takes on as its focal task the study of *non-standard* worlds. These fall into two major groups: *schematic* worlds for which principle (I) fails, and *inconsistent* worlds for which principle (II) collapses. A schematic world will be such that, for some theses, neither they nor their negations obtain. The world is *indeterminate* with respect to these theses. An inconsistent world will be such that certain theses obtain together with their negations. Such worlds are *inconsistent* with respect to these theses—they embrace incompatible alternatives. (The combination of these cases can of course also arise.) We shall endeavor to trace out the major consequences of such a less restrictive approach to ontology, one which envisages a broadening of our ontological horizons beyond the traditional confines of the standard possible worlds.

It is necessary to insist at the very outset that one should avoid speaking of *inconsistent* worlds as *impossible* worlds. This would be question-begging, for it is a prime aim of the present analysis to show that they can be considered as genuinely possible cases. It is the manifestation of a prejudice—well entrenched but by no means correct—to hold that inconsistent worlds are *ipso facto* impossible. Inconsistent objects and worlds, so we shall argue, are feasible targets of rational consideration and scrutiny. They too can be meaningfully assumed, supposed, hypothesized, etc. And the supposition of such worlds is emphatically not an invitation to logical chaos. One can reason perfectly cogently and coherently about them. (Thought need not bear the attributes of its objects: a sober study of inebriation is altogether feasible, and so is a consistent study of inconsistency.)

To be sure, inconsistent and schematic worlds may well seem strange, unaccustomed, even bizarre. For the actual world—this familiar world of ours—is (presumably) a standard one which is (hopefully) devoid of inconsistencies or incompletenesses. But even if this is indeed so, an insistence that this feature of our world must inevitably characterize worlds in general has something parochial about it. A feature that characterizes our own *actual* world need surely not characterize all *possible* worlds whatsoever.

A world-description can be viewed as making an assertion with claims certain things to be the case. In admitting inconsistent worlds we ourselves are accordingly safeguarded from *self*-contradiction because (as it were) *another* assertor—the world-description at issue—is effectively introduced as intermediary between us and the contradiction. It is the world at issue that is inconsistent, but not necessarily our own discourse about it. (We ourselves do not declare both P and $\sim P$, but only $T_w(P)$ and $T_w(\sim P)$.) In mooting the prospect of inconsistent worlds, one thus takes a position that is—or can be—perfectly cogent and consistent within itself.[4]

The avoidance of inconsistency unquestionably presents an important regulative principle for the claims of our own discussions. It is, no doubt, desirable to avoid inconsistency in our own thought and our own assertions. But it is by no means equally imperative to shun the recognition of possible inconsistencies in the *objects* of this thought and assertion. (Thought—to reiterate the point—need not share the features of its objects.) The consistent theoretical scrutiny of inconsistent worlds is not only an attainable but perhaps even a useful goal.

SECTION 3

An Ontological Perspective

The present discussion proposes to be utterly serious about non-standard worlds. As mentioned above, our perspective is that the non-standardness at issue is to be *ontological*—that it characterizes the world itself, rather than reflecting a merely epistemic concern with our knowledge, information, or beliefs about this world.

Thus in the case of a schematic world, the situation is not just that *we* don't know whether *P* or its contradictory $\sim P$, but that *the world itself is indeterminate* in this regard in its make-up—it is ontologically indecisive in point of *P* vs. $\sim P$. A schematic world is a total blank in this regard. (It is, to put it figuratively, just unable to make up its mind about *P* vs. $\sim P$.) It is as though a mischievous Cartesian demon had taken a standard world and made erasures in it, so that the world is blank or fuzzy in certain respects. The situation is one of *ontological underdetermination*—with regard to certain envisageable states of affairs the world is simply "incomplete". Worlds introduced by way of hypotheses are a standard example: In saying: "Suppose that there were an elephant in yonder corner . . ." one remains uncommitted as to its exact size, type, sex, etc.[5] Schematic worlds have ontological blanks or blurs which leave them underdefined in certain specifiable respects. This prospect seems especially plausible in a framework of *emergent* properties within a situation of temporal development which produces a succession of different "worlds" (or world-states) in such a way that the successive transitions move matters from the more to the less schematic. (That is, the earlier worlds are schematic with respect to properties, dispositions, or laws that only appear on the scene later on.) This perspective is posed by the sort of evolutionism from simple to more complex worlds or world-states envisaged by Herbert Spencer and C.S. Peirce in the latter part of the 19th century.[6]

The situation with respect to inconsistent worlds is roughly analogous. The case here is not merely that of *cognitive* dissonance—of having excellent grounds for accepting some thesis *P* but also other, equally meritorious grounds for accepting $\sim P$. Rather, the world is *ontologically overdetermined.* It is inconsistent—there is some state-of-affairs claim *P* such that both *P* and $\sim P$ obtain in it. Such worlds are overdefined in point of specification, embodying a conflation of discordant elements. The situation of such ontological dissonance is something like an over-printing of discordant pictures or a text designed to equivocate between divergent claims. (It is as though there were superimposed realities, as with the conflation or superposition of distinct action-patterns on a television screen.)[7] The make-up of the world embodies a synthesis or fusion of incompatible states of affairs.[8] This can be thought of as taking a number of individually and separately altogether self-consistent worlds, and ramming them together into a composite superimposition whose descriptive make-up—albeit inconsistent—does not go beyond the content of its consistent constituents.

The ontological situation in a world w can accordingly reflect two basic situations with regard to a given state-of-affairs-purporting thesis P:

(i) P obtains; its ontological status is "on," symbolically: $[P]_w = +$.

(ii) P fails (to obtain); its ontological status is "off", symbolically: $[P]_w = -$.

Here $[P]_w$ is to represent the ontological status—as existent or inexistent in the world w—of the state of affairs purported by P. And we shall suppose that in *this* regard the ontological situation is orthodoxly two-valued ($+$ or $-$) according as P obtains or fails; it is "on" or "off." We may thus adopt the idea of a $(+/-)$ register to indicate the on/off status of every proposition.

The crucial facet of the stance we are taking here is that the obtaining-status ($+$ or $-$) of P and that of $\sim P$ in a possible world are to be independent of one another. The cardinal doctrine of the analysis is that *the ontological status of P and that of* $\sim P$ *should be seen as strictly independent issues.* The existential status of a P-positive state of affairs is to be decoupled from that of a P-negative one: the facticity of P is not to preclude that of $\sim P$.

Accordingly there will be four distinct possibilities for the pair $[P]_w$, $[\sim P]_w$, which we will symbolize by $\{P\}_w$:

$[P]_w$	$[\sim P]_w$	$\{P\}_w$
$+$	$+$	$++$
$+$	$-$	$+-$
$-$	$+$	$-+$
$-$	$-$	$--$

With non-standard worlds we have the anomalous circumstance that P and $\sim P$ can *both* receive $+$, or can *both* receive $-$. In the nice—standard or classical—worlds, to be sure, only the combinations $+-$

or $-+$ can occur. But the critical point is that this decisiveness is now seen not as an inevitable feature of *all* worlds but merely as the characteristic defining feature of certain sort of world (to be sure, a particularly important one).

It is crucial to note, however, that while we may well have it that $[P]_w = [\sim P]_w = +$ in some inconsistent world w, we shall certainly never have it that $[P \,\&\, \sim P]_w = +$. Our stance is that two mutually inconsistent states of affairs might well both be realized in a nonstandard world, whereas a single *self inconsistent* state of affairs can never be realized. Contradictions can be realized distributively but not collectively: *self-contradiction* must be excluded. We shall always have $[P \,\&\, \sim P]_w = -$.

Consequently, the assignment of the ontological $(+/-)$-values cannot behave in a "truth functional" way *vis-à-vis* the usual propositional connectives. Specifically, one cannot say—as one might well be tempted to—that $(+ \,\&\, +) = +$. (For consider the case that $[P]_w = [\sim P]_w = +$. Then $[P \,\&\, P]_w = +$ whereas $[P \,\&\, \sim P]_w = -$.)

The independence of $[P]_w$ and $[\sim P]_w$ thus has far-reaching implications. For it means that logical principles cannot take us far in drawing inferences about the $(+/-)$-status of theses on the basis of information about the $(+/-)$-status of their logical congeners. As we have just seen, it is not even generally the case that:

If $[P]_w = +$ and $[Q]_w = +$, then $[P \,\&\, Q]_w = +$.

Moreover, the fact that $[P]_w$ and $[\sim P]_w$ might both be $+$ means that we will *not* have:

If $[P]_w = +$ and $P \vdash \sim Q$, then $[Q]_w = -$.

(Here $\vdash$ represents ordinary—i.e., classical—deductive consequencehood.) For when $Q = \sim P$, we might very well have it that $[Q]_w = [\sim P]_w = +$, while yet $[\sim Q]_w = [P]_w = +$.

However, we do indeed have the *Principle of Immediate Inference:*

If $[P]_w = +$ and $P \vdash Q$, then $[Q]_w = +$.

In no circumstances could this most fundamental of logico-ontological principles be compromised. It is built into the very meaning of the relationship $\vdash$ of (classical) logical deducibility.

Someone might ask: "Why characterize the nonstandardness of inconsistent worlds in terms of the concurrent obtaining of *contradictories*, why not in terms of the concurrent obtaining of *contraries*?"

The answer is that it does not matter. For it is readily shown that the former case includes the latter. Thus suppose that:

(1) P and Q are contraries; i.e., $P \vdash \sim Q$
(2) In the world w, both P and Q obtain, i.e., both $[P]_w = +$ and and $[Q]_w = +$.

Then from $P \vdash \sim Q$ and $[P]_w = +$ we have it by the Principle of Immediate Inference that $[\sim Q]_w = +$. And this, together with $[Q]_w = +$, puts us into a non-standard situation along the previously contemplated lines, one in which contradictories obtain concurrently.

SECTION 4
Two Modes of World-Fusion: Schematization and Superposition

But exactly how is one to conceive of non-standard worlds: can they be conceptualized coherently?

Let us deal with this issue by indicating two concrete procedures for constructing coherent descriptions of such worlds, procedures whose task is to provide a coherent and logically viable mechanism for conceptualizing these logically anomalous worlds.

Let the w_i be worlds—*standard* (classical, orthodox) worlds, in the first instance. Then we may introduce the procedure of

World-conjunction (schematization)

$w_1 \frown w_2$ is that world such that, for any proposition P, P obtains in this world iff P obtains *both* in w_1 *and* in w_2:

$$[P]_{w_1 \frown w_2} = + \text{ iff } [P]_{w_1} = + \text{ and } [P]_{w_2} = +$$

And so the ontological status of any proposition in the $\frown$-world can be determined in terms of its status in the constituent worlds with reference to the following table:

$[P]_{w_1}$	$[P]_{w_2}$	$[P]_{w_1 \frown w_2}$
+	+	+
+	−	−
−	+	−
−	−	−

Conjunction-worlds are in general underdetermined or schematic. In such $\frown$-worlds it will sometimes happen that neither P nor $\sim P$ obtains. For suppose $[P]_{w_1} = +$ and $[P]_{w_2} = -$. Then $[P]_{w_1 \frown w_2} = -$. But $[\sim P]_{w_1} = -$ and $[\sim P]_{w_2} = +$, since we may suppose that the "internal logic" of the w_i is classical. And so $[\sim P]_{w_1 \frown w_2} = -$. Hence neither P nor $\sim P$ obtains in $w_1 \frown w_2$; this world is incomplete, as it were, in point of P.

Such schematic worlds are, accordingly, ontologically fuzzy or blank or "illegitimate" in their make-up: certain of their features are "lost in the fog" of their indeterminacy. For example, let w_1 contain three individuals *a, b, c* and let w_2 contain the two individuals *a, d*. Moreover, let the relation R be such that it obtains in w_1 only between *a* and *b*, and in w_2 only between *a* and *d*. Then since $\exists x(aRx)$ obtains both in w_1 and w_2, it will also obtain in $w = w_1 \frown w_2$. Thus the property of "being R'd by *a*"—that is, $\lambda y(aRy)$—will characterize some individual of w. But it will *not* hold of any particular, *identifiable* individual of this world. (Indeed, the only *identifiable* individual member of w is a, for a is the only value of y such that $\exists x(x = y)$ is true both in w_1 and w_2; but nevertheless "There are at least two individuals" will hold in w, since it holds for both w_1 and w_2.) Thus w will contain a shadowy—and in some respects "nondescript"—individual, and it is of this individual (say ξ) that $aR\xi$ obtains.

Consider another example, let the membership of w_1 be the integers 2, 3, 4 and that of w_2 be 5, 6, 8. Then we can say of the population of $w_1 \cap w_2$ that it contains exactly one integer of the set $\{3, 5\}$ and exactly two of $\{2, 4, 6, 8\}$, since each of these contentions hold for both w_1 and w_2. (Thus $\cap$ and the $\cap$ of set-theoretic intersection operate quite differently.) We can specify one of the items at issue as "the odd member" of $w_1 \cap w_2$ and say a good deal about it—but not anything that would distinguish between 3 and 5.

Schematic worlds have a foggy sector in whose penumbral darkness there are somewhat strange goings-on. This is something we must take in our stride if we are going to deal with schematic worlds at all.[9]

Furthermore let us also introduce the procedure of

World-disjunction (superposition)

$w_1 \cup w_2$ is that world such that, for any proposition P, P obtains in this world iff P obtains *either* in w_1 *or* in w_2:

$$[P]_{w1 \cup 2w} = + \text{ iff } [P]_{w_1} = + \text{ or } [P]_{w_2} = +$$

Thus the ontological status of any proposition in the $\cup$-world can be determined in terms of its status in the constituent worlds with reference to the following table:

$[P]_{w_1}$	$[P]_{w_2}$	$[P]_{w_1 \cup w_2}$
+	+	+
+	−	+
−	+	+
−	−	−

Disjunction-worlds are in general *overdetermined*. In $\cup$-worlds it can happen that both P and $\sim P$ obtain. For suppose $[P]_{w_1} = +$ and $[P]_{w_2} = -$. Then $[P]_{w_1 \cup w_2} = +$. But $[\sim P]_{w_1} = -$ and $[\sim P]_{w_2} = +$, since (as we have, *pro tem*, supposed) the "internal logic" of the w_i is classical. And so $[\sim P]_{w_1 \cup w_2} = +$. Thus here both P and $\sim P$ will obtain in $w_1 \cup w_2$.

The ontological status of propositional claims in conjunction worlds or disjunction worlds can be worked out mechanically in terms of a simple four-valued "logic" in the case of two worlds.[10]

Four status-values will be operative in a compositive world $w_1 * w_2$, as follows:

$[P]_{w_1}$	$[P]_{w_2}$	$[\![P]\!]_{w_1 * w_2} (= \langle [P]_{w_1}, [P]_{w_2} \rangle)$
+	+	+ +
+	−	+ −
−	+	− +
−	−	− −

Consider now the following four-valued tables:

$[\![P]\!]$	$[\![\sim P]\!]$	$[\![P]\!]$ \ $[\![Q]\!]$	$[\![P \,\&\, Q]\!]$				$[\![P \vee Q]\!]$			
			+ +	+ −	− +	− −	+ +	+ −	− +	− −
+ +	− −	+ +	+ +	+ −	− +	− −	+ +	+ +	+ +	+ +
+ −	− +	+ −	+ −	+ −	− −	− −	+ +	+ −	+ +	+ −
− +	+ −	− +	− +	− −	− +	− −	+ +	+ +	− +	− +
− −	+ +	− −	− −	− −	− −	− −	+ +	+ −	− +	− −

This multivalued system is in fact a well-known system of many-valued logic. It is the logical system corresponding to the many-valued truth tables denominated as Group I in C.I. Lewis' Appendix II to *Lewis and Langford* (1932). This system was much touted in a later paper by Jan Lukasiewicz ("A System of Modal Logic," *The Journal of Computing Systems*, vol. 1 [1953], pp. 111–149). Actually it is simply the Cartesian product of classical two-valued logic with itself.[11]

These tables can be used to work out the value-assignment of compound theses in a routine and mechanical fashion. Thus to determine the ontological status of a thesis in a $\Cap$-world in terms of its status in the base worlds, one simply uses the rule: $++$ goes to $+$, all others to $-$. And in a $\Cup$-world one uses the rule: $--$ goes to $-$, all others go to $+$. This makes the determination of the $(+/-)$-status of theses in con- or dis-junction worlds a matter of routinized calculation (i.e. recursion).

Consider, for example, the case of the two (standard) microworlds specified by the following property-allocative description tables with respect to the same two properties and the same two individuals:

	w_1		w_2	
	F	*G*	*F*	*G*
a	−	+	−	+
b	+	−	−	+

Consider now the question of the status of $Fa \vee Gb$ in $w_1 \Cap w_2$ and $w_1 \Cup w_2$. Note that:

$$- - \vee - + = - +$$

But $-+$ goes to $-$ in $\cap$-worlds and to $+$ in $\cup$-worlds. Accordingly, the thesis in view has a different ontological status in the two sorts of worlds at issue.

Note, however, that while we can move inferentially from the $(+/-)$-status of a proposition in separate worlds to their $(+/-)$-status in a fusion world, this process cannot be reversed. For example, if we have $[P]_{w_1 \cap w_2} = -$, then we do not know whether the $(+/-)$-status of P is $-$ in w_1 only or in w_2 only or in both. Information can become lost in fusion: we may know *that P* fails to obtain in the fusion world, but now *how.*[12]

The $\cup$-process for constructing inconsistent worlds through the "composition" of standard ones affords the basis for making a crucially important point. Such inconsistent worlds are *not* open to the charge of leading to the catastrophe of a logical chaos in which *anything* can be asserted as true. (Assertion is, of course, only a meaningful prospect in contexts where not everything can be asserted—where there is some exclusion. For assertion must be determinative, and *omnis determinatio est negatio* ["Every determination is a negation"], as Spinoza's precept has it.) Inconsistent worlds will still, in general, exclude many possibilities. As long as we hold steadfast to the simple and perfectly intelligible principle that what holds in such an inconsistent superposition world is neither less nor more than what holds somewhere within the specified base worlds (which themselves are presumably of an impeccably orthodox nature), we can rest assured of avoiding the disaster of logical chaos. It is emphatically not the case that "anything goes" in such inconsistent worlds—that any and every claim whatever, no matter how wild, obtains in them. This avoidance of logical chaos is, of course, crucial for the prospect of maintaining the viability and potential utility of such worlds.

Brief consideration must be given to the issue of the *individuation* of non-standard worlds. In individuating standard worlds, it suffices to know the [ ] status of the elementary (or *atomic*) propositions at issue. With non-standard worlds we have to be more subtle and involve the entirety of their { } status. But this { } status of the propositions at issue does indeed suffice to settle the issue of world-individuation.

Thus in the two-parameter case, we obtain sixteen possible worlds, with twelve non-standard additions to the four familiar standard ones, as follows:

p	$\sim p$	q	$\sim q$	*Comment*
+	+	+	+	an inconsistent world
+	+	+	−	an inconsistent world
+	+	−	+	an inconsistent world
+	+	−	−	an inconsistent *and* schematic world
+	−	+	+	an inconsistent world
+	−	+	−	a standard world
+	−	−	+	a standard world
+	−	−	−	a schematic world
−	+	+	+	an inconsistent world
−	+	+	−	a standard world
−	+	−	+	a standard world
−	+	−	−	a schematic world
−	−	+	+	an inconsistent *and* schematic world
−	−	+	−	a schematic world
−	−	−	+	a schematic world
−	−	−	−	a (hyper)schematic world

Consider the manifold of standard possible worlds spanned by the three (independent) atomic propositions p, q, and r:

	p	q	r
w_1	+	+	+
w_2	+	+	−
w_3	+	−	+
w_4	+	−	−
w_5	−	+	+
w_6	−	+	−
w_7	−	−	+
w_8	−	−	−

By $\cup$-combinations we will obtain possible worlds that fuse several of these elementary worlds into a superimposition world. Thus p obtains unproblematically in $w_1 \cup w_2 \cup w_3$, where as q and r both have an anomalous status in that both they and their negations have a + status in this world. This world, albeit inconsistent, is clearly nevertheless possibility-excluding, seeing that $\sim p$ is emphatically false in it. Note further that, since possible worlds are individuated in terms of the obtaining-status or $(+/-)$-categorization of the relevant proportions, we must (for example) identify $w_1 \cup w_2 \cup w_3$ with $w_1 \cup w_4$, as both corresponding to the case $\{p\} = +-$, $\{q\} = ++$, $\{r\} = ++$.

Analogously, by $\cap$ combinations we will obtain possible worlds that conjoin several of these elementary worlds into a schematic world. Thus p obtains (and similarly $\sim p$ fails) unproblematically in $w_1 \frown w_2 \frown w_3$, whereas this world is a blank with respect to q and r, which are such that both they and their negations have a $-$ status in this conjunctive world. However $w_1 \frown w_2 \frown w_3$, although uncommitted with respect to q and r, counts as a genuine possible world, seeing that it is committed with respect to *something* (e.g., p vs. $\sim p$). Again note that we must identify $w_1 \frown w_2 \frown w_3$ with $w_1 \frown w_4$, since we wish to individuate possible worlds on the basis of the obtaining-status or $(+/-)$-categorization of the relevant propositions, and both cases have it in common that $\{p\} = +-$, $\{q\} = --$, $\{r\} = --$.

Non-standard worlds of this sort do not thrust themselves upon us by an ineluctable force of their own: they are the products of our devising—we can construct their descriptive make-up as we please. We ourselves are in control of their defining characterization, and we can manipulate this descriptive make-up as we wish. It is, after all, *our* own hypothesis that is the characterizing basis of such worlds: we ourselves occupy "the driver's seat" here. And what we wish to realize is a compound world in which *just exactly that* obtains which obtains in *each* (or else *in some one*) of a series of base worlds. This is the guiding principle of the conceptualization of such worlds, and there is nothing in itself infeasible or impossible about it, short of a dogmatic insistence that "A *world* could not possibly be like that!"[13]

SECTION 5

Logical Inference and Semantical Status

In considering the standing of semantical principles in our extended theory of possible worlds, it is convenient to introduce the truth-operator along the lines of the equivalence:

$t_w(P)$ iff $[P]_w = +$

This straightforwardly formalizes the traditional thesis that truth is *adaequatio ad rem*—that in any possible world the truth must agree with (its) reality and that a true assertion is one that "corresponds with the facts."

This truth-operator facilitates the setting out of semantical principles for non-standard worlds.

To begin with we have it that the characterizing principles of our of our non-standard conjunction-worlds and disjunction worlds assume that

$t_{w_1 \cap w_2}(P)$ iff $t_{w_1}(P) \wedge t_{w_2}(P)$ ('$\wedge$' for "and")
$t_{w_1 \cup w_2}(P)$ iff $t_{w_1}(P) \vee t_{w_2}(P)$ ('$\vee$' for "or")

It may be noted that all of the following traditional semantical principles will now obtain:

(1) $\vdash P \Rightarrow t_w(P)$, for all w. ('$\Rightarrow$' for "if . . . then")
(2) $t_w(P), P \vdash Q \Rightarrow t_w(Q)$, for all w.
(3) $t_w(P \,\&\, Q) \Rightarrow t_w(P)$ [and also $t_w(Q)$], for all w.

Note, however, that—as we have already in effect seen—all of the following classical semantical principles must be *rejected* when non-standard worlds are brought upon the stage of consideration:

(4*) $t_w(\sim P) \Rightarrow \neg t_w(P)$ ('$\neg$' for "not")
(5*) $\neg t_w(P) \Rightarrow t_w(\sim P)$
(6*) $t_w(P)$ and $t_w(Q) \Rightarrow t_w(P \,\&\, Q)$ [fails for inconsistent worlds only]
(7*) $t_w(P)$; $t_w(Q)$; $P, Q \vdash R \Rightarrow t_w(R)$
(8*) $t_w(P \vee Q) \Rightarrow t_w(P)$ or $t_w(Q)$ [fails for schematic worlds only]

One of the salient inferential principles of traditional logic is the conjunction principle:

(CP) $P, Q \vdash P \,\&\, Q$

It might appear on first view that this inferential principle clearly fails in the context of inconsistent worlds. For in *such* a world w we may well have both P and $\sim P$, but will never have $P \,\&\, \sim P$.

Actually, however, what fails for non-standard worlds is not this principle (CP)—or indeed *any* principle of logic itself—but rather the semantical meta-principle connecting the inferential principles of logic with the semantical issue of the truth-status of propositions in possible worlds. The line of thought at issue here pivots about the following

Fundamental Rule of Valid Inference

(R) Whenever $P_1, P_2, \ldots, P_n \vdash Q$ is a valid inference principle of classical logic, and $t_w(P_1)$, $t_w(P_2)$, . . . , $t_w(P_n)$, then $t_w(Q)$.

This rule holds for all *standard* worlds. But it fails for non-standard worlds, as is shown by the following example:

w_1 corresponds to $p \,\&\, \sim q$
w_2 corresponds to $\sim p \,\&\, q$
$w = w_1 \cup w_2$

Note that with respect to w we have both $t(p)$ and $t(q)$. Yet not $t(p \,\&\, q)$. The implication $t(p), t(q) \Rightarrow t(p \,\&\, q)$ clearly fails, despite the fact that (CP) obtains unproblematically.

Two *distinct and importantly different constructions* of the principle "valid inferences from true premisses yield true conclusions" are available. There is, first of all, a *distributive* reading—"from *distributively* true premisses"—as per the rule (R) above. This, as we have seen, can fail in non-standard worlds. But there is also a *collective* reading—"from *collectively* true premisses" as per the rule:

(R′) Whenever $P_1, P_2, \ldots, P_n \vdash Q$ is a valid inference principle of classical logic, and $t_w(P_1 \,\&\, P_2 \,\&\, \ldots \,\&\, P_n)$, then $t_w(Q)$

And *this rule will always obtain*, even in non-standard worlds, thanks to the Principle of Immediate Inference of Section 3 above. (With $\cap$-worlds there can—to be sure—be no difference between collective and distributive truth, but with $\cup$-worlds the case is emphatically otherwise.) With non-standard, and specifically with inconsistent worlds, the Fundamental Principle of Valid Inference "If the premisses of a valid inference are true, then the conclusion must also be true" obtains *only* in the form (R′). Here it is crucial to give the collective (not distributive) reading to the "true premisses" clause of the fundamental principle.

Someone might well object: "But what exactly is '$\vdash$' to *mean* if you abandon (*R*)?" The answer is simple: the deduction-sign '$\vdash$'

continues to stand for what it has *always* stood for—an airtight guarantee that from the (supposed) truth of the premisses taken together we can infer that of the conclusion. The only change is that we now resolve the equivocation or ambiguity between the constructions "the premisses *taken collectively*" and "the premisses *taken distributively*"—and resolve it in favor of the former.

Ordinarily, there is—to be sure—no difference between (R) and (R′), for throughout all *standard* worlds the distributive truth of theses and their collective truth simply coincides. But this equivalence does *not* obtain in non-standard worlds—specifically in inconsistent ones. (Here—as we have just seen—one can have $t(P)$ and $t(Q)$ without $t(P \& Q)$.)

Accordingly, one can retain classical logic holus bolus, even in the context of non-standard worlds, *provided* that one goes over to a non-classical semantics by replacing (R) by (R′). The crucial fact is simply that the classical equivalence of

$$P_1, P_2, \ldots, P_n \vdash Q$$

with

$$t(P_1), t(P_2), \ldots, t(P_n) \Rightarrow t(Q)$$

must be abandoned.

The difference between (R) and (R′) turns pivotally on the unavailability of the principle: $t(P)$, $t(Q) \Rightarrow t(P \& Q)$. In information-processing contexts there is a substantial difference between the conjunction $P \& Q$ and the juxtaposition P,Q—a difference which it is well worthwhile to heed and to preserve in our information-processing operations. This is brought out by such examples as the following:

(1) Conjunction and juxtaposition may be separated to preserve the distinction of sources. If Source No. 1 gives us p and Source No. 2 maintains q, then we have p,q. But we should not automatically take ourselves to be in possession of $p \& q$; nobody (as yet) has vouched for it. Thus getting $r \& {\sim}r \& s$ from one (obviously confused) source is from the informative point of view quite a different sort of thing from getting the conflicting reports r and ${\sim}r \& s$ from two sources.

(2) Contexts of probable and inductive reasoning require the distinction between conjunction and juxtaposition as well. If p and q are both individually probable (or both "inductively indicated"), it by no means follows that $p \& q$ is so. And

inductive considerations can powerfully substantiate p,q without thereby substantiating $p \& q$.

The failure of this adjunction principle in *epistemic* contexts (and especially in *inductive* contexts) is widely recognized.[14] Rational acceptance lacks the feature of being "logically closed" in that one can be warranted in accepting both P and Q, but not in accepting their conjunction $P \& Q$.[15] Our present stance is that this principle must be given up for *alethic* contexts as well, once one's possible-world semantics envisages the prospect of inconsistent worlds. (A bridge can be built between the two positions by taking the stance that we really have no *a priori* guarantee that the world we endeavor to know on the epistemic side is in fact a consistent one on the side of its ontology.)

An adjunction principle can take distinct forms, the following three in particular:

(A) as the deductive principle:
$$P, Q \vdash P \& Q$$

(B) as the semantical principle:
$$t(P), t(Q) \Rightarrow t(P \& Q)$$

(c) as the metatheorematic principle:
$$\vdash P, \vdash Q \Rightarrow \vdash P \& Q$$

Our present approach rejects (B) alone, retaining (A) and (C) intact. We have to do with an unorthodox *semantics*, not an unorthodox *logic*. Moreover, the contrast between (A) and (B) forcibly reminds us of the importance of the ancient Aristotelian distinction between an *absolute* (unmodalized) proposition and a proposition in the modality of actuality: we must resist any temptation to identify P and $t(P)$ (although possibly, and perhaps presumably, the *actual* world w^* will meet those conditions—especially consistency—under which a Tarskian principle P iff $t_{w*}(P)$ can be postulated).

Most inconsistency-tolerating logics—the systems of *relevant* implication,[16] for example—try to avert disaster in the presence of inconsistency by blocking the classical inferential move from the incompatible pair P, $\sim P$ to an arbitrary thesis Q. Our present tactic, however, is to accept $P \& \sim P \vdash Q$, but to block the move from the two mutually contradictory theses P and $\sim P$ to their self-contradictory conjunction $P \& \sim P$. This blockage is not—to be sure—accomplished within *logic* itself (for we shall retain (A)), but rather in the *semantical* setting of any world-relative or system-relative assertion (as per (B)).[17]

Of course, the converse of the adjunction principle (B), namely $t(P \& Q) \Rightarrow t(P), t(Q)$ obtains unproblematically. But the original principle will require a special additional proviso as to the mutual *cotenability* of P and Q. (In the context of mutually consistent and compatible theses, we need have no reservations about this semantical adjunction principle—it becomes problematic only in the inconsistent case.) In general, however, the tenability of a thesis P (in one context) and that of Q (in another) does not establish the tenability of their *conjunction*, $P \& Q$, because P and Q might fail to obtain in one and the same context, so that their separate tenability does not suffice to assure their cotenability. And our ontological posture is that an inconsistent world might include two distinct but *mutually* inconsistent states of affairs, but that a single *self-consistent* circumstance simply cannot qualify as a "state of affairs" capable of inclusion in a possible world.

Ordinarily, in orthodox and standard cases, separate tenability in one and the same world is a sufficient basis for cotenability,[18] and hence suffices to assure the validity of (B) throughout such cases. But with the introduction of non-standard (and specifically *inconsistent*) worlds the idea that co-presence in one selfsame world assures cotenability must be abandoned. And this abandonment leads to the collapse of a proviso crucial to the acceptability of (B).

This failure of the orthodox semantical adjunction principle $t(P), t(Q) \Rightarrow t(P \& Q)$ has the far-reaching consequence that any and every inference rule which involves the combining of distinct premisses (i.e., any rule that deploys a series of premisses distributively) must itself be treated in its light of these considerations. For example, consider the *modus ponens* principle:

(MP) If $t_w(P)$ and $t_w(P \supset Q)$, then $t_w(Q)$.

This principle also fails, as is shown by the following example:

$$w_1 = p \ \& \sim q$$
$$w_2 = \sim p \ \& \sim q$$
$$w = w_1 \uplus w_2$$

Note that with respect to w we have $t(p)$, since p obtains in w_1, and also $t(p \supset q)$, since $p \supset q$ (that is, $\sim p \vee q$) obtains in w_2. But we do not have it that q obtains in w, since q fails to obtain in both of the w_i.

Observe, however, that the cognate principle

(MP′) If $t_w(P \ \& \ [P \supset Q])$, then $t_w(Q)$

is perfectly acceptable even for non-standard worlds (inconsistent worlds, in particular). It is a straightforward consequence of the Principle of Immediate Inference.

And the same story holds, *mutatis mutandis*, for other multi-premissed inferential principles, all of which must be construed in a collective rather than distributive manner.

The fact that separate tenability does not automatically guarantee cotenability introduces a certain complexity into the semantics of nonstandard worlds.

There is something inherently non-truth-functional about such worlds: there is no general way of giving the truth-in-*w* conditions of a molecular statement in terms of the truth-in-*w* conditions of its components (save in the case of those special nonstandard worlds which we know *a priori* to be fusions of standard ones).[19] There are two ways of looking at this—either as a decisive blockage to the viability of such worlds, or as a recognition of the difficult facts of life outside the sphere of standardist simplicity. There is no need to argue at length the merits of the latter perspective.

SECTION 6

Does Inconsistency Produce Logical Anarchy?

A world w exhibits a *singularity of overdetermination* at P iff both $t_w(P)$ and $t_w(\sim P)$, and it exhibits a *singularity of underdetermination* at P iff neither $t_w(P)$ nor $t_w(\sim P)$.

The crucial issue is the question of whether semantical singularity can be contained or whether it will be pervasive, whether like a virulent cancer it metastasizes so that its presence in one place betokens its inescapable presence throughout. (Cp. Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics*, V.8.). It is a key thesis of our present analysis that semantical singularity can be a *local* phenomenon that does not invariably have *global* ramifications —that the occurrence of a singularity *somewhere* does NOT entail its recurrence *everywhere*.

On first sight it appears that this situation must be otherwise—at any rate if we accept a classical logic. For if we accept both of the following inferential principles,

(I) $P \vdash P \vee Q$
(II) $P \vee Q, \sim P \vdash Q$

then it would seem that a contradiction at one place immediately becomes all-pervading. For consider the following chain of reasoning:

(1)	$t_w(P)$	by assumption
(2)	$t_w(\sim P)$	by assumption
(3)	$t_w(P \vee Q)$	from (1) and (I) by the Principle of Immediate Inference
(4)	$t_w(Q)$	from (2), (3), (II)

This argument purports to show that the acceptance of a single anomaly—as with the combination of (1) and (2)—leads inexorably to the truth of any arbitrary thesis Q, as per (4).

But the argument is fallacious. It fails at step (4). This step collapses, since its use of (II) rests on a "collective-truth" reading regarding $\sim P$ and $P \vee Q$, whereas the premisses (2) and (3) only yield the materials for a "distributive-truth" reading. With non-standard worlds, as we have seen, one does not have the classical rule of inference (R) of the preceding section, but only its weaker counterpart (R′). And so the proposed line of vitiating argument cannot in fact be carried through. Singularities need *not* be all-pervasive—and therefore all-destructive.

Similarly, it is generally held that a contradiction-tolerating system must abandon the classical thesis

$$(X) \vdash (P \,\&\, {\sim}P) \supset Q$$

embodying the principle *ex [per necessitatem] falso quodlibet*, lest a single inconsistency produce logical chaos. But to exploit this thesis to establish an *arbitrary* thesis Q on the basis of the contradictory combination both P and ${\sim}P$ in some world w, one would need to reason essentially as follows:

(1) $t_w(P)$	by assumption
(2) $t_w({\sim}P)$	by assumption
(3) $t_w(P \,\&\, {\sim}P)$	from (1), (2) since $R, S \vdash R \,\&\, S$
(4) $t_w(Q)$	from (3) and (X) by the Principle of Immediate Inference

But this chain of reasoning collapses at step (3), because of its mistaken use of the Fundamental Rule of Valid Inference, which in these contexts obtains only in its *collective* version, and not in the *distributive* version that is employed here.

Given such restrictions, it is emphatically not the case that a contradiction anywhere diffuses into a contradiction everywhere—*not even if classical logic is retained.* It is mistaken to hold that a contradiction-admitting theory is impossible within the framework of classical logic (though, to be sure, we must here utilize the rather subtle distinction between collective and distributive truth).

The key point of these considerations is that the rules of logic do not fail for non-standard worlds as such, but only as regards their *semantical* exploitation in relating the truth-status of inferentially related theses. The introduction of non-standard worlds requires an unorthodox semantics, but induces no *logical* matters: our logic can continue to be altogether classical.

Someone might well propose the following consideration:

> Your logic of non-standard worlds is not really classical at all, because the connectives are not the classical ones. Conjunction, for example, is not your & but is really given by the good old familiar formula:
>
> $t_w(P \oplus Q)$ iff both $t_w(P)$ and $t_w(Q)$.
>
> Only this rigidly truth-functional connective is real conjunction,

> not the essentially intentional connective you are dealing with. And the same goes, *mutatis mutandis*, for the other propositional connectives.

This opens up an important line of consideration. For it is clear that, in the context of non-standard worlds, the conjunction-connective $\oplus$ behaves in an unorthodox (i.e. non-classical) way (e.g., the inference from $P \oplus \sim P$ to Q is no long valid). In sum, the recognition of non-standard worlds forces us to realize that in construing the "essential character" of a logical connective we must make a priority-choice between its *logic* (the inferential principles it underwrites) and its semantics (the truth-determination principles it generates). Only with standard worlds do these two issues go hand in hand. In general the less auspicious conditions of non-standard worlds constrain us to the recognition that these two contexts (logic and semantics) pull in separate directions, and that we must *choose* along which of these lines our connectives are to be construed—whether inferential or truth-deterministic considerations are to be determinative of "real" conjunction, disjunction, etc. Our own choice here is clear—we follow the mainstream of logical tradition in giving priority to the inferential aspect, taking the stance that what a logical connective "really is" is to be determined in terms of what it *does* in inferential situations.

SECTION 7

Modes of Inconsistency

The idea of tolerating inconsistencies can be construed in the following four ways (arranged in order of increasing unpalatability):

(A) [Weak Inconsistency] To accept the prospect that for some genuinely possible world *w*:

$t_w(P)$ and $t_w(\sim P)$, for some *P*.

(B) [Strong Inconsistency] To accept the prospect that for some genuinely possible world *w*:

$t_w(P \,\&\, \sim P)$, for some *P*.

(C) [Hyperinconsistency] To accept the prospect that for some genuinely possible world *w*:

$t_w(P \,\&\, \sim P)$, for all *P*.

(D) [Logical Chaos] To accept the prospect that for some genuinely possible world *w*:

$t_w(P)$, for all *P* (and accordingly $t_w(P)$ and $t_w(\sim P)$, for all *P*)

The presently envisaged tolerance of inconsistency does not extend beyond case (A). We continue to share the general (and doubtless right-minded) abhorrence for cases (B)–(D). The weak inconsistency whose ontological prospects we are prepared to recognize is emphatically not all-embracing: it does not spread like a logical cancer into strong inconsistency or beyond—into logical chaos. Case (A) need not be construed as leading inexorably to (D). Inconsistency can represent a *local* and not necessarily *global* anomaly. It is not necessarily pervasive and vitiating, but may prove to be sporadic and harmless (like a "singularity" in mathematics).

But could one not press a tolerance of inconsistency beyond case (A)? To see how the matter stands here, consider the supposition that a world is inconsistent in sense (B) (which the others entail). Only one rationally viable upshot now seems possible. While cogent considerations might move us to accept the truth of *P* and (concurrently) that of $\sim P$—in that "there is much to be said on both sides"—surely nothing could (rationally) move us to accept an outright SELF-contradiction of the form $P \,\&\, \sim P$. The point is at bottom an epistemic one—no cogent line of consideration could ever move us to accept such a contention, and a world-picture in which this circumstance is projected can thus serve no workable ontological purpose. We have little choice but to regard the very hypothesis we are making as self-destructive, as simply annihilating itself. A world picture must clearly satisfy certain constraints of intelligibility if it is to count as depicting a possible world and a blatant self-contradiction fails in this respect. Logical truths aside, an intelligible assertion must rule something out (it must determine something and *omnis determinatio est negatio*), and a self-contradiction fails us in this regard. (Logical truths are *trivial*—they obtain in all possible worlds; self-contradictions are assertively futile in exactly the opposite respect because, *ex vi terminorum*, in view of the conceptual content of the very idea at issue, they can be true in no possible world at all—they fall outside the range of the practical politics of ontological deliberation.) In uttering an outright self-contradiction we literally "cannot say" what we have it in mind to assert.

The correct view on this point is surely that of an essentially Kantian perspective. We must hold that *minimal consistency*—the exclusion of any but the weakest form of inconsistency—is not a *constitutive* feature of the world (not, that is, ultimately a properly descriptive characteristic of it) but a *regulative* feature, an indispensable aspect, that is, of our very *concept* of a world. Here the demands of intelligibility would constrain us to suppose that it is not the world at issue that is self-contradictory (and unintelligible) but the very assumption that we are being asked to make about it. Minimal consistency should be viewed as a condition of rational intelligibility—of the way in which we must needs transact our conceptual business if the controlling interests of logical viability are to be achieved. The insistence on minimal consistency—and in general the idea of reducing inconsistencies to the smallest convenient[20] proportions—is a crucially important regulative principle of cognitive systematization.

On such a view, its minimal consistency (i.e. not being more than *weakly* inconsistent) is not a constitutive and descriptive feature of the world, but is, in the final analysis, a regulative and conceptual feature of our understanding of it, an aspect—to put it bluntly—not of *reality* as such but of our procedures for its conceptualization and accordingly of *our conception* of it. To say of reality *per se* that it is "incoherent"—that it is inconsistent in some strong sense of the term—would be to say of it something that, in the final analysis, is simply meaningless. The hypothesis that reality is incoherent is ultimately senseless because to suggest of reality that it might actually meet this condition is to ask of it something that is in principle unrealizable. At the metaphysical level the thesis that reality is more than minimally inconsistent is essentially empty because no stable sense can possibly be made of its denial. Weak inconsistency, however, is something else. Here (as we have seen) there are various proposals for making sense of inconsistency on a local anomaly basis. On such a limited basis, an Averroistic "double-truth" theory that countenances both $t_w(P)$ and $t_w(\sim P)$, for the propositions P of a certain limited family, can be entertained without driving us across the boundaries of unintelligibility.

SECTION 8

More on Semantics

The semantics of non-standard worlds rests on the basis of the following stipulation regarding $/P/_w$, the truth-value of the proposition P with respect to the world of w:

$/P/_w = T$ iff P obtains in w; that is, iff the P-entry in the $(+/-)$-register for this world w is $+ : [P]_w = +$.

The well-known *Tarski Principle* "a that-P claim is true iff P" must be rejected by any inconsistency-tolerant logic in the form $t(P)$ iff P. For this would at once transmute $t(P) \wedge t(\sim P)$ into the wholly untenable $P \,\&\, \sim P$. However, the principle continues operative in the specific form:

$t_w(P)$ iff $/P/_w = \mathrm{T}$ iff P obtains in w iff $[P]_w = +$.[21]

These equivalences yield the immediate consequences:

In $\cap$-worlds, truth behaves conjunctively:

$$/P/_{w_1 \cap w_2} = T \text{ iff } /P/_{w_1} = T \text{ and } /P/_{w_2} = T$$

In $\cup$-worlds, truth behaves disjunctively:

$$/P/_{w_1 \cup w_2} = T \text{ iff } /P/_{w_1} = T \text{ or } /P/_{w_2} = T$$

Note the important upshot that *the orthodox (classical) logical truths remain true in ALL possible worlds, even the nonstandard ones.* In fact, we can unproblematically continue to endorse the Leibnizian stipulation that the necessary truths are exactly those true in all possible worlds. The propositional logic inherent in a possible world semantics embracing nonstandard worlds is such that we arrive at the following situation:

(1) *All* of the valid theses of classical logic will obtain in *all* possible worlds (non-standard ones included).

(2) The *only* propositions that obtain in *all* possible worlds (non-standard ones included) are the valid theses of classical logic.

The acceptance of non-standard worlds does *not* force any sort of strange logic upon us.

The truth-value of falsity, F, may be accommodated in our semantics through the rule:

$/P/_w = F$ iff $\sim P$ obtains in w; that is, iff the $\sim P$-entry in the $(+/-)$-register for w is $+$:

$[\sim P]_w = +$.

Accordingly:

$$/P/_w = \begin{cases} \text{T iff } [P]_w = + \\ F \text{ iff } [\sim P]_w = + \end{cases}$$

We thus obtain the traditional equivalences that $/P/_w = \text{T}$ iff $/\sim P/_w = \text{F}$ and also $/P/_w = \text{F}$ iff $/\sim P/_w = \text{T}$. In these respects, negation continues to behave classically, even after the admission of non-standard worlds. However, we also have it that $/P/_w$ is *undefined* whenever $\{P\}_w = --$. Thus F and T are no longer mutually exhaustive. One must abandon the Law of Excluded Middle for schematic worlds (as was remarked at the outset). Again, whenever $\{P\}_w = ++$ we have it that both $/P/_w = \text{T}$ and $/P/_w = \text{F}$—the truth-value assignment is no longer single-valued. One must thus abandon the Law of Contradiction for inconsistent worlds, as we have seen.

Note that:

(1) Both $/P/_w = T$ and $/\sim P/_w = T$ can hold with respect to an inconsistent world w (one in which $+$ can occur for both $[P]$ and $[\sim P]$). And this is so despite fact that inevitably (for all P and w):

$/P \,\&\, \sim P/_w = F$

(2) Neither $/P/ = T$ nor $/\sim P/ = T$ may hold with respect to a schematic world w (one in which $-$ can occur for both $[P]$ and $[\sim P]$). And this is so despite the fact that inevitably (for all P and w):

$/P \vee \sim P/_w = T$

While it can happen in non-standard worlds that there are theses for which $/P/_w$ and $/\sim P/_w$ are *both* T or *both F*, this can only transpire when P is a contingent proposition, never when P is an (orthodox) logical truth or a negation thereof.

Thus classical principles like the "Law of Excluded Middle"

For all w: $t_w(P \vee \sim P)$

or the "Law of Contradiction"

For all w: $\neg t_w(\sim P \,\&\, \sim P)$

continue operative in their object-language version. And so do their *strict* metalinguistic counterparts:

For all w: $t_w(P) \vee \neg t_w(P)$
For all w: $\neg[t_w(P) \wedge \neg t_w(P)]$

But, as we have just seen, their *mixed* counterparts

For all w: $t_w(P) \vee t_w({\sim}P)$
For all w: $\neg[t_w(P) \wedge t_w({\sim}P)]$

will fail when non-standard worlds are brought within our purview. When *inconsistent* worlds (in particular) enter upon the scene, we have to reckon with the prospect that:

For some w: Both $t_w(P)$ and $t_w({\sim}P)$, for some P.

If the Law of Contradiction is so construed as to entail rejection of *this* prospect, then our present doctrines fall afoul of it. But if the law is construed differently, as precluding

For some w: $t_w(P \;\&\; {\sim}P)$, for some P

then our theory has no quarrel with this law. (Again, the difference between distributive and collective truth proves crucial.)

However, while the *strict* metalinguistic counterparts of the classical logico-semantical principles are *always* demonstrably correct principles on the present account, the *semantical metalogic* of non-standard worlds is, nevertheless, definitely non-classical and indeed radically unorthodox. As we have seen, with inconsistent worlds we do not even have the conjunction principle

$$t_w(P),\ t_w(Q) \Rightarrow t_w(P \;\&\; Q)$$

or the principle of *modus ponens* detachment:

$$t_w(P),\ t_w(P \supset Q) \Rightarrow t_w(Q)$$

Such metalogical principles—as we now recognize—will not hold for non-standard worlds in these distributive-truth versions, but only in their conjunctively variant collective-truth versions. It is not all that difficult to operate metalogically with non-standard worlds, but one must watch carefully what one is doing.

Thus consider the schematic world $w_1 \frown w_2$ based on two standard microworlds:

w_1		w_2	
$[p]$	$[q]$	$[p]$	$[q]$
+	−	−	+

Note that, since $p \vee q$ obtains in both microworlds, we have $t_{w_1 \frown w_2}(p \vee q)$. Yet neither $t_{w_1 \frown w_2}(p)$ nor $t_{w_1 \frown w_2}(q)$. Thus we cannot in general make the inference:

$$t_w(P \vee Q), \neg t_w(P) \Rightarrow t_w(Q).$$

The rejection of such a version of the "Law of Excluded Middle" is also part of the unorthodoxy of the semantics of non-standard worlds.

Again, another striking aspect of this unorthodoxy is the failure of the substitutional interpretation of quantifiers. That is, $\exists x \phi x$ may be true in w, and yet $\phi\alpha$ fail to be true for every *identifiable* individual α of the domain at issue. This can be seen as follows: Consider the schematic world $w_1 \frown w_2$ obtained from the following two (*ex hypothesi* standard) microworlds: (1) w_1 based on individuals a, b such that $t_{w_1}(\phi a)$ and $t_{w_1}(\phi b)$, and (2) w_2 based on the individuals a, c such that $t_{w_2}(\bar{\phi} a)$ snd $t_{w_2}(\phi c)$. Then since $\exists x \phi x$ is true in both w_1 and w_2 it is also true in $w_1 \frown w_2$. But note that there is only one *identifiable* individual of $w_1 \frown w_2$, namely a. (For a is the only value of y for which $\exists x(x = y)$ obtains in $w_1 \frown w_2$.) And we do *not* have $t_{w_1 \frown w_2}(\phi a)$. The substitutional interpretation of quantifiers accordingly does not provide a workable semantics for the quantification theory of non-standard worlds.[22]

SECTION 9

Meinongian Objects

The machinery of non-standard worlds provides a natural and efficient device for articulating a theory of non-standard objects along the lines envisaged by Alexius Meinong (1853–1920).[23] (It should be understood that our discussion here is revisionary rather than historical—we are doing Meinong-*reconstruction* and not Meinong-*exegesis*. The distinctions at issue in our discussion were obviously not available to Meinong himself, and their use must thus be interpreted as affording effective means towards Meinongian ends rather than as representing the means by which he himself proposed to attain those ends.)

Consider the following two (standard) microworlds answering to the indicated descriptions, based on two properties, *F* and *G*, and two individuals, *a* and *b*:

w_1

	F	$\bar{F}$	G	$\bar{G}$
a	+	−	+	−
b	−	+	−	+

w_2

	F	$\bar{F}$	G	$\bar{G}$
a	+	−	−	+
b	+	−	−	+

These stipulations lead to:

$w_1 \dot{\cap} w_2$

	F	$\bar{F}$	G	$\bar{G}$
a	+	−	−	−
b	−	−	−	+

$w_1 \dot{\cup} w_2$

	F	$\bar{F}$	G	$\bar{G}$
a	+	−	+	+
b	+	+	−	+

This example illustrates the following general facts:

(1) The individuals of $\dot{\cap}$-worlds are in general descriptively underdetermined: there will be cases in which *neither* ϕx *nor* $\bar{\phi} x$ obtain.

(2) The individuals of $\dot{\cup}$-worlds are in general descriptively overdetermined: there will be cases in which both ϕx and $\bar{\phi} x$ obtain.

Such an approach straightforwardly underwrites the idea of schematic and of inconsistent individuals.

The possession of properties is now governed by the semantic principle:

x has ϕ (in w) iff $[\phi x]_w = +$ iff $/\phi x/_w = T$

This leads to the result that two quite different situations can arise:

x lacks ϕ (in w) iff $/\phi x/_w \neq T$ iff $\{\phi x\}_w$ is $-+$ or $--$.
x has $\bar{\phi}$ (in w) iff $/\bar{\phi} x/_w = T$ iff $\{\phi x\}_w$ is $-+$ or $++$.

The lack of a property and the possession of its complement are in principle different: in non-standard worlds we can have one without the other.

In schematic worlds we may have neither ϕx nor $\bar{\phi} x$. In such worlds there can be Meinongianly "incomplete (or *indeterminate*) objects" of which neither ϕ nor $\bar{\phi}$ can be (truly) predicated, objects whose property-structure is only partially defined.

And in inconsistent worlds we can have Meinongianly "inconsistent objects" for which both ϕx and $\bar{\phi} x$. A "Law of Contradiction" of the form

$\forall^w x \sim(\phi x \ \& \ \bar{\phi} x)$

where a world-relative universal quantifier is at issue, will hold only with respect to certain possible worlds (viz., consistent ones), but will not obtain in general.[24]

Again, in inconsistent worlds we may have both ϕx and ψx, for two incompatible properties ϕ and ψ, as with Meinong's round, square object. But note that the conjoining of ϕx with $\bar{\phi} x$ à la ϕx-and-$\bar{\phi} x$ is something very different from maintaining $(\phi\text{-and-}\bar{\phi})x$. (For as we know, $t_w(\phi x) \wedge t_w(\psi x)$ does *not* entail $t_w(\phi x \ \& \ \psi x)$.) A world might contain an object that is "a round square" in the sense of being round *and* being square, but it cannot contain a "round-square," on the present approach. The inconsistent properties have to be asserted of an object not in one breath, but in two, so to speak.

To characterize *inconsistent* objects as *impossible* objects clearly begs some crucial questions.[25] But if we are to view them as even potentially "possible", then we must construe this as different from *logical* possibility in the sense:

possible = *logically* possible = self-consistent
= compatible with classical logical principles of object-description.

Instead, we must adopt the construction:

possible = *semantically* possible = describable (or "conceivable" or "thinkable") = descriptively constructible = compatible with semantical principles.

In sum, we must envisage the realm of the coherently discussable as broader than that of logically self-consistent (at any rate in the sense of classical logic).

Fiction is clearly a source of non-standard objects. Fictional items can be schematic, as Hamlet's hat size or shoe size clearly represent indeterminacies. Or again, fictional objects may be inconsistent, as with the fateful letter of Anthony Trollope's novel *Ayala's Angel.*[26]

From this Meinongian perspective, it becomes a moot question whether it makes sense to contemplate a "*totally indeterminate* object," one that is altogether schematic with respect to *every* (non-trivial) property, a "bare particular," a *mere* thing or being; let us call it $\mathscr{E}$ (for *ens*). Such a property-bereft item is reminiscent of the "indeterminate being" of Hegel's *Logic*, of which he says—no doubt rightly—that it is tantamount to *non-being*. To assume such a "thing" to be a *thing*—an individual—presumably makes no sense. Accordingly, $\mathscr{E}$ falls under the ax of the medieval logician's principle: *nihil sunt nullae proprietates* ("Nothing has no properties at all").[27]

Moreover, we now have the prospect of a "*totally overdeterminate* object," one that possesses *every* property (and thus its complementary negation as well), let us call it $\mathscr{M}$ (for *utter mishmash*). Again, the assumption that a "thing" so characterized can be a *thing* indeed—an at any rate *possible* individual—presumably makes no sense. There must always be the prospect of making an informative—i.e., determinate—statement about any genuine thing. Accordingly, $\mathscr{M}$ falls under the ax of Spinoza's principle: *omnis determinatio est negatio*.

No doubt, a reasonable principle of definiteness might be introduced to restrict the scope of Meinongian objects; they must all presumably both have and lack some properties—in the classical mode of possession. It seems plausible to suppose that there is no thing where there can in principle be no viable *description* of it, and a viable description must on the one hand *include* some properties (thus blocking $\mathscr{E}$) and on the other hand *exclude* some (thus blocking $\mathscr{M}$). An acceptance of incomplete and impossible Meinongian objects thus need not be construed as going so far as to commit us to qualifying $\mathscr{E}$ and $\mathscr{M}$ as genuine individuals.

These brief indications may perhaps suffice to show how a reconstruction of Meinong's *Gegenstandstheorie*[28] can be effected rather straightforwardly within the framework of a theory of non-standard possible worlds.[29]

SECTION 10

Paradoxes

Our considerations regarding non-standard worlds bear interestingly upon the logical paradoxes. Consider, for example, the version of the Liar Paradox based on the statement:

L = "This statement is false" (or, equivalently, "The contradictory of this statement is true").

This engenders the situation that L is true iff it is false:

$$/L/ = T \text{ iff } /L/ = F \text{ (iff } /{\sim}L/ = T)$$

Returning now to the basic semantical rule that

$$/P/_w = \begin{cases} \text{T iff } [P]_w = + \\ \text{F iff } [{\sim}P]_w = + \end{cases}$$

we see that this amounts to:

$$[L] = + \text{ iff } [{\sim}\text{L}] = +.$$

Thus L can indeed obtain, but, since we will have $/L/ = T$ iff $\{L\} = ++$, it follows that L can do so only in an inconsistent world. However, given the prospect of such a world, there is no need to reject L as *meaningless* (≅ "lacking in truth value") because its realization-as-true is now an available alternative. The paradoxical thesis at issue is smoothly accommodated, in a perfectly viable way, within the resources of a non-standard possible world.

Note, however, that the case is otherwise with respect to

L' = "This statement is not true."

For this engenders the situation that L' is true iff it is also not true:

$$/L'/ = T \text{ iff } /L'/ \neq T$$

or equivalently

$$[L'] = + \text{ iff } [L'] \neq + \text{ iff } [L'] = -.$$

And this is not possible relative to the ontological assumptions postulated above to the effect that the ontological [ ]-status of a thesis is *always* defined, so as to comport itself classically. On this basis, the revised Liar Paradox thesis would have to be rejected as improper and meaningless—on our semantical theory as on the standard one. Special eliminative devices—along the lines of rejection as "meaningless"—will thus still have to be deployed to avert the revised Liar Paradox generated by L'. The availability of non-standard worlds does not of itself automatically resolve all of the difficulties posed by semantical paradoxes.

It is worthwhile to pose the following question:

> In L we have a statement whose accommodation as assumptively true forces recourse to an *inconsistent* world. Is there an analogous paradoxical statement whose truth-conditional accommodation forces recourse to a *schematic* world?

The answer here is negative. For suppose that there were such a statement S. Then we would have

$$/S/ = T \text{ iff } C(S)$$

for some suitable condition C whose force would, *ex hypothesi*, be such as to constrain the schematic case. But then this C would have to assure that both $[S] = -$ and $[\sim S] = -$. However $/S/ = T$ already calls for $[S] = +$. This circumstance indicates the unrealizability of the required condition C.

Throughout the orbit of the logical paradoxes—both semantical (e.g., Grelling's Paradox of the ontological/heterological) and mathematical (e.g., Russell's Paradox or Burali Forti's Paradox)—one encounters a generic situation whose shared structure is as follows. It emerges that, in some suitably contrived context, one is led by straightforward and intuitively cogent reasoning from acceptable-seeming premisses to the conclusion that both P and $\sim P$ (for some appropriate thesis P). Historically, the discovery of such a paradox has invariably met with the reaction that we must abandon or somehow abridge the plausible and straightforward line of reasoning used in extracting the anomaly at issue. And this approach has not been particularly successful. As one recent writer summarizes the situation:

> Of course, we know how to avoid the paradoxes formally. We can avoid the semantic paradoxes, e.g. by a hierarchy of Tarskian metalanguages, and the set theoretic ones, e.g. by the class/set distinction of von Neumann. But these are not solutions. A paradox is an argument with premises which appear to be true and [inferential] steps which appear to be valid, but which nevertheless ends in a conclusion which is false. A *solution* would tell us which premise is false or which step invalid; but moreover it would give us an *independent reason* for believing the premise or the step to be wrong. If we have no reason for rejecting the premise or the step other than that it blocks the conclusions, then the "solution" is *ad hoc* and unilluminating. Virtually

all known "solutions" to the paradoxes fail by this test and this is why I say that no solution has yet been found.[30]

However, the mechanism of non-standard worlds affords us a resource for another, very different course of approach. We can simply *accept* the paradox, treating it as a situational singularity—an isolated difficulty, a local anomaly that has no wider global ramifications. That is, we might be led to a *restricted* tolerance of inconsistency by the usual force of the cost-benefit parameters of formalization: expository adequacy, systematic efficiency, convenience in applications, simplicity, etc. We would take (isolable) inconsistencies in stride if doing so enables us to realize substantial systematic advantages elsewhere.

It may be useful to illustrate the workings of this approach in the particular case of Russell's Paradox.

The Russell paradox is posed by a certain set, namely that introduced by its definition as the set of all sets that are not elements of themselves: the set **R** of all sets x such that $x \notin x$. Its definitional specification is designed to introduce the set **R** as governed by two operative conditions:

(1)$\forall x(x \in \mathbf{R} \supset x \notin x)$

(2)$\forall x(x \notin x \supset x \in \mathbf{R})$.

The pivotal question now arises: Is **R** or is **R** not an element of itself? Do we have $\mathbf{R} \in \mathbf{R}$ or $\mathbf{R} \notin \mathbf{R}$. Either way we are in trouble:

(i) If $\mathbf{R} \in \mathbf{R}$, then by (1) $\mathbf{R} \notin \mathbf{R}$; and therefore $\mathbf{R} \notin \mathbf{R}$ by *reductio ad absurdum*.

(ii) If $\mathbf{R} \notin \mathbf{R}$ then by (2) $\mathbf{R} \in \mathbf{R}$; and therefore $\mathbf{R} \in \mathbf{R}$ by *reductio ad adsurdum*.

On the present inconsistency-embracing line of approach, we are enjoined to grasp the nettle and take this situation in stride. We are simply to accept the fact that our set theory has a singularity at **R**. Think of an analogy. Just as a Meinongian world can contain a non-standard object *a* for which both *Fa* and $\bar{F}a$, so a sufficiently enterprising set theory can envisage a non-standard or anomalous set **S** for which both $x_1 \in \mathbf{S}$ and $x_1 \notin \mathbf{S}$ (for some suitable x_1).[31] Accordingly, the upshot of the Russell paradox is simply that the set **R** happens to be anomalous with respect to self-membership—i.e., that both $\mathbf{R} \in \mathbf{R}$ and $\mathbf{R} \notin \mathbf{R}$. The "Russell set" **R** is simply a Meinongian object which (demonstrably) both has certain property and its contradicting complement as well. (Considering Russell's own uncomprehending dismissal of Meinong's views, there is a charming historical irony—and an element of poetic justice—to this fact that Russell's paradox can be resolved by this device of classing the Russell-set as a Meinongian inexistent *Gegenstand.*)

A set theory that is singularity-tolerant in envisaging the prospect of such anomalous sets is not thereby shown to be unworkable, let alone absurd.

The general lessons of this line of thought can be exfoliated in somewhat fuller detail in the light of these considerations.

The logical and mathematical paradoxes arise because a certain group of basic assumptions $A_1, A_2, \ldots, A_n$ conjointly yield both X and $\sim X$ (for a certain thesis X). Closer analysis of the situation indicates that assumption A_i is the prime source of the trouble. Now the usual and orthodox stance is that A_i must be abandoned or amended.[32] But on the present approach we can retain A_i and merely *proceed to identify it as the gateway to a merely LOCAL region of anomaly within a fundamentally viable global framework.* From this perspective, the unrestricted principle of set-formation of naive set-theory (say) does not need *replacement*—it need merely be acknowledged as demarcating a restricted region of problematic claims.

Such an approach can be pushed one step further. The usual and orthodox procedure would be to endeavor to *replace* A_i by an A^*_i in such a way that the resultant replacement-set is once more consistent. On the present line of approach, however, we would endeavor to divide A_i into two separately viable, individually unproblematic components

$$A_i = (A^1_i \mathbin{\&} A^2_i)$$

in such a way that the replacement of A_i within the A_j-set by either of these two would produce a mutually consistent result. Such a resolution would enable us to view the initial assumption-set $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ as the (*collectively* inconsistent) superposition of the two discordant assertion-zones at issue. Our set-theory becomes an inconsistent whole with local irregularities located in the environment of problematic axioms whose problematic impact could be removed by a suitable splitting. On this approach, the proper treatment of inconsistency proceeds by way not of the revision of an inconsistency-producing axiom, but of its division. This possibility renders the inconsistency harmless because localizable—a mere irregularity in a theory that is otherwise perfectly workable (its overall inconsistency notwithstanding).

Such a strategy is not a matter of mindlessly disregarding the inconsistency that is encountered. The inconsistency is not ignored but taken seriously, as representing an anomalous circumstance, but one which is now seen as calling for a different diagnosis and treatment: a process of division rather than replacement. No less ingenuity and inventiveness is needed for this superpositionist treatment than is required on the traditional revisionist strategy, but this is now expended in a different direction—that of pinpointing clearly just at what points the anatomy of the axiom system exhibits divisions that permit the necessary splits to be made.

It becomes important in this context to distinguish clearly between logical truth as such (which we have symbolized as per the turnstile operator $\vdash P$) and demonstrability in a formal system **S** based on certain assumptions (which we may symbolize as per $|\!\vdash_S P$). This latter must be treated in terms of assertion-as-true within the "system" ($\cong$possible world) at issue, and thus treated as per $t_S(P)$. That this system relative assertion must not be assimilated to $\vdash$ in the case of an inconsistent system is shown by the fact that while we would certainly wish to retain the rule

$$\vdash P, \vdash Q \Rightarrow \vdash P \,\&\, Q$$

we would equally certainly wish to avoid

$$|\!\vdash_S P, |\!\vdash_S Q \Rightarrow |\!\vdash_S (P \,\&\, Q)$$

which would clearly lead to disaster in the case of an inconsistent formal system **S**. Intra-systemic demonstrability or assertability is to be viewed as something quite different from strictly *logical* demonstrability or assertability as such, and is to be assimilated to holding in an "assertion-zone" that is the analogue of a possible world.

Applying this general line of approach to the Russell paradox we note that this can be construed as showing that both members of the pair of **R**-characterizing theses (1) and (2) above—respectively $\forall x(x \in \mathbf{R} \supset x \notin x)$ and $\forall x(x \notin x \supset x \in \mathbf{R})$—will not be true together in the *same* assertion zone of the set-theoretic sphere, since their pairing produces a singularity of overdetermination. Thus we can have $t[(1)]$ and $t[(2)]$ distributively, but not the collective $t[(1) \,\&\, (2)]$. We must construe our set theory as composed of two assertion-zones Z_1 and Z_2, in such a way that the whole is simply their union, and that each of (1) and (2) is true in just exactly one of the zones.[33] Thus while these theses are *distributively* true, we must not construe this to yield their *collective* truth. The critical step is that of avoiding the conjunctive concatenation of theses across the boundaries of different assertion zones. This tactic, as we have seen, enables us to tolerate inconsistent theses, accepting distributively a conjunction of theses which, accepted collectively, will produce disaster. In this way inconsistencies can be accepted without engendering untoward consequences.

Think of an analogous situation in the theory of Meinongian objects. Suppose that:

(A) in w_1, a has the property $\phi = \lambda x \forall y Rxy$ ("*a R*'s everything")
(B) in w_2, b has the property $\psi = \lambda x \sim \exists y Ryx$ ("nothing *R*'s *b*")

Then in $w_1 \cup w_2$, we have it that *a R*'s everything AND nothing *R*'s *b*. One can take two approaches here: (i) the orthodox line that $w_1 \cup w_2$ simply does not qualify as a "possible world," or (ii) the presently envisaged line that $w_1 \cup w_2$ does indeed qualify as a "possible world," albeit an inconsistent one in which both ϕa and $\bar{\phi} a$ and also both ψb and $\bar{\psi} b$. And if we grasp the nettle of the second alternative, we simply have to conceive of the resultant inconsistent world as the fusion of two severally consistent assertion zones, that of w_1 whose point of entry is ϕa, and that of w_2 whose point of entry is ψb, which leads to an incompatible destination. Paradox, in sum, is seen as merely a fusion of discordant elements that involves localizable conflicts—a fusion that need not be taken to be all-destructive, but merely to have untoward consequences whose damage can be isolated and contained.

Accordingly, our policy can be one of *accepting* the paradox, rather than endeavoring to *eliminate* it by abandoning one or another of those several plausible and natural-seeming contentions which figure in its emergence. For example, when one considers the logical and mathematical paradoxes, for instance, it is often striking how much more straightforward, plausible, and intuitive a line of reasoning leads into the paradox, as contrasted with the unnatural, controversial, and *ad-hoc*-seeming devices that are proposed as means of extracting us therefrom. The "solutions" usually offered for the semantical antinomies and the mathematical paradoxes are always such as to exact an enormous price in the introduction of complex, cumbersome, and intrinsically unappetizing machinery. The acceptance of inconsistency—once it becomes rationally feasible—enables us to realize those very virtues of systematic simplicity whose pursuit is an essential aspect of the cognitive enterprise.

Such a paradox-embracing approach does not (or need not) deny that paradox represents an anomaly, an undesirable condition of things, a negative factor—in sum, a *liability*. But what it denies is that paradox must be logically *fatal*, that it is a *decisively* negative consideration, an end-all. Inconsistency becomes a merely local anomaly that is, to be sure, disadvantageous but not absolutely catastrophic. It is viewed as a finite misfortune that can be tolerated to avoid yet greater misfortunes, but not an irretrievable disaster that must be avoided at all costs.

The traditional thesis of the *consistency of nature* might also be regarded in this light. The considerations canvassed here suggest that such a principle need no longer be seen as a matter of logico-conceptual necessity. The picture of our world as self-consistent is one alternative among others—an alternative to be adopted on the essentially *empirical* grounds of what sort of theory of nature best serves the systematic needs of our inquiry, rather than on the basis of the abstract "general principles" of the matter. The issue becomes an empirical—and thus an essentially *contingent*—matter. (To be sure, "empirical" does not mean *observational* here, but *experiential* in the far broader sense of a theoretical triangulation from observation—a systematization of the products of inquiry.)[34]

Given *this* stance towards paradox, its occurrence is seen not as an end of the matter—a closing of the book of logical deliberation. Rather, it becomes an invitation to embark upon a cost-benefit analysis. One can treat paradox as a matter of *local* inconsistency or singularity, an occasional difficulty, but not a pervasive disaster. It is possible—nay perhaps even probable—that the tolerance of such a liability might be offset by advantages elsewhere. It becomes worth examining whether the negativity of paradox might perhaps be offset by the positivity of systematic advantages attaching to the procedures that permit them—the avoidance of complications, systematic simplicity and convenience, comprehensiveness of content, uniformity of approach etc.[35] It thus becomes a matter of comparative cost-benefit analysis to see whether the negativity of paradox-acceptance might not in fact be counterbalanced by certain systematic advantages over its paradox-avoiding rivals. (In the case of the Russell paradox, for example, there is no question that all of the familiar stratagems designed to block this antinomy exact a substantial price in the espousal of intrinsically implausible suppositions, in themselves without natural and intuitive appeal, and introduced simply and solely owing to their *ad hoc* capacity to circumvent the paradox.)[36]

Note that from this perspective it also becomes possible to give a new and altogether different reading of the import of Kurt Gödel's celebrated demonstration of the incompletability of arithmetic.[37] For Gödel showed that one can always—within any system of arithmetic articulated by the usual recursively axiomatic processes—formulate an arithmetical truth which, in effect, asserts its own unprovability. To have a *complete* axiomatization of arithmetic that captures all its truths, one would accordingly need to prove such a theorem—and thereby automatically convict the system at issue of inconsistency. Traditionally, this forced choice between completeness and consistency is construed—in the light of an acceptance of consistency as an absolute and wholly non-negotiable requirement—to show that axiomatic arithmetic is uncompletable and inherently incapable of capturing the whole sphere of arithmetical truth. But the present perspective indicates the possibility of a very different interpretation, namely that there can be no fully adequate axiomatic formalization of arithmetical truth outside the domain of non-standard (inconsistency-admitting) systems—that *if arithmetic (in toto) is to be systematized at all in the usual recursively axiomatic way, then this must be done by means of a paradox-admitting, non-standard system.*[38] This upshot clearly places the significance of this sort of system into a prominent and very favorable light.

Again, one can also take a very different view of Alfred Tarski's well-known argument that natural language is rationally defective and logically unworkable because it is inconsistent—its truth-conditions being such that one is forced to class mutually inconsistent propositions as true. From our present perspective, this view of the matter grossly exaggerates the seriousness of the situation. Provided (as is certainly the case) that the "inconsistency" of natural languages can be confined to isolable pockets thereof—and relatively small ones at that—we need not take the stance that we must jettison this invaluable resource to preserve our pretentions to logical seriousness.

To be sure, there is no blinking the fact that our present line of approach to inconsistency toleration also exacts a price—it commits us to an unorthodox semantics. But this is a price we pay once and for all. We are not forced to make a repeated series of *ad hoc* concessions in set theory, in arithmetic, in natural philosophy, in the theory of language, etc.

Seen in this light, the development of inconsistency-tolerant systems has a close kinship with mathematical catastrophe theory. For our concern is to analyse and accommodate what happens "when things go wrong"—when those singularities occur which indicate that, from the standpoint of the usual order of things something has blown up on us, as it were. In this respect we may view the logic of inconsistency as a functional equivalent of catastrophe theory in logic in its effort to handle gravely anomalous situations in a systematic and logically cogent way.

SECTION 11

The Ideology of Inconsistency

A recognition of contradictions in nature goes back to the Presocratics.[39] And if not Hegel himself,[40] then at any rate many of his followers maintained the realization of contradictions in the world.[41] Marxists of various sorts are nowadays strident supporters of such a view. It is a major historic position that merits careful evaluation.

The consistency of nature is certainly no fact of experience. Quite the reverse! As the ancient sceptics stressed, experience confronts us with an inconsistent world: sight tells us the stick held at an angle under water is bent, touch tells us it is straight. Each eye presents a somewhat different picture of the world: the brain alone enables us to "see" it consistently. To assert the consistency of nature is to express one's faith that the mind will be able ultimately to impress consistency upon the results of our inquiry. But this confidence may in the final analysis prove to be misplaced. It is at best a *hope* that all such *apparent* discords are merely that and admit of ultimate reconciliation, a hope for whose realization a good deal of theorizing is required. And there is no reason of *a priori* general principle to think that this will always prevail—that we might in fact not always be able to push the state of theorizing out to a point where all awkward conflicts admit of smooth theoretical reconciliation.

Consistency is an epistemic desideratum—but not one that is absolute and unqualified. Man's mind does not thrive on consistency alone: the blockage to order that results from conflicting images is a crucial goad to inquiry and a pivotal motive for enlarging our information. Intellectual disequilibrium is a powerful constructive force. A critical aspect of man's evolutionary success lies in the fact that the central nervous system of higher animals demands expectation-contravening inputs to avoid boredom.[42] And the actual course of human inquiry undoubtedly exhibits the structure of a St. Simonian ebb and flow of alternation between synthetic eras of consolidation and analytic eras of conflict, of (relative) cognitive dissonance and cognitive harmony. Inconsistency plays no less important role on the stage of our cognitive affairs than does consistency.

But could a suitable rationale of justificatory warrant ever be found to induce a rational man to accept a view of the world as actually inconsistent? There is surely some reasonable prospect of it. For suppose an epistemic situation in which:

(a) there are compelling reasons to think that one of two mutually incompatible alternatives P_1 and P_2 must obtain,

(b) there are compelling reasons against suspending judgment on this issue (reasons of a theoretical rather than practical sort—having to do with our capacity to arrive at an adequate and rationally satisfying systematic view of "the true nature" of things). A failure to take a stance with respect to the P_1/P_2 question would represent a decisive impediment in the way of our efforts to "engross the truth" and develop a workable theory of the phenomena under consideration,

(c) there is no available rational basis for deciding between these two alternatives, and not just this, but it can be shown convincingly that one cannot expect to find a rational basis for deciding between them.

It can now surely be argued—and not implausibly argued—that in circumstances of this sort we are entitled to accept both P_1 and P_2 *provided* that "we can make a rational go of it," that is, provided that the systematic structure of the situation is such as to create no intolerable strains in the face of this anomaly.

It is often held that consistency is an indispensable requisite of cognitive rationality—that the toleration of *any* inconsistency, no matter how minor or local it may appear to be, is simply *irrational.* This contention is misguided.

The keystone of cognitive rationality is the idea of doing as well as we possibly can in the cognitive enterprise—of *optimizing* our attainment of its defining objectives. The maintenance of consistency—desirable though it be—must be subordinated to this ruling *telos.* This point can be brought home by an example. Consider the circumstance of a choice between two situations: (1) we have only a small area of knowledge—only a handful of our key questions are effectively answered—but there are no contradictions, vs. (2) we have a substantial area of knowledge—a great many of our key questions are effectively answered—but there is one small, localized region of inconsistency. Confronted by such a choice, is there anything irrational about preferring a situation whose structure is of the second kind to one whose structure is of the first? Surely not!

Consistency, important though it be, is but one desideratum among others: it is not a be-all and end-all. There is nothing irrational about its toleration in appropriate circumstances—i.e., when the realization of other cognitive desiderata outweighs the negativity it unquestionably represents. The prospect of inconsistency-toleration is not to be aborted by invoking irrationality as bogeyman.

Our desire to "capture the truth" may well (and rightfully) *overpower* our wish to "avoid inconsistency." There may conceivably be circumstances in which a cognitive vacuum is intolerable and must be avoided even at the price of inconsistency.[43] Consistency is a merit—an unquestionably substantial merit, and not just the hobgoblin of little minds.[44] But it is a merit that can be purchased at too dear a price. And the price is too dear if its payment blocks the path to inquiry, impeding our endeavors to get at the truth of things.

One (admittedly strictly epistemological) route to an inconsistent picture of things is afforded by epistemic probabilism, which bases the acceptance of theses on their standing in point of probability. For if we take probability as our "guide of life" in cognitive matters, and espouse theses on this basis, we will be led straightaway into inconsistency. This is illustrated by the so-called Lottery Paradox of inductive logic. This paradox is the immediate result of a decision-policy for acceptance that is based upon a probabilistic threshold value. Thus let us suppose the threshold level to be 0.80, and consider the following series of statements:

This (fair and normal) die will not come up i when tossed ($i = 1, 2, \ldots, 6$).

According to the specified standard, it transpires that every one of these six statements must be accepted as true. Yet their conjunction results in a patent absurdity.[45] Moreover, the fact that the threshold was set as low as 0.80 instead of 0.90 or 0.9999 is wholly immaterial. To recreate the same problem with respect to a higher threshold we need simply assume a lottery wheel having enough (equal) divisions to exhaust the spectrum of possibilities with individual alternatives of sufficiently small probability. Then the probability that each specific result will *not* obtain is less than 1 minus the threshold value, and so can be brought as close to 1 as we please. Accordingly we should, by accepting each of these claims, be driven to the impossible conclusion that there is no result whatsoever.

This, then, is the Lottery Paradox. It is often construed in discussions of inductive logic as showing that a probabilistic criterion of acceptance must be rejected as unworkable.[46] But as the context of our present discussion shows, it could instead be viewed to show that this epistemic standard is actually not untenable, but merely anomalous in yielding the picture of the world as an inconsistent possible world. The person whose epistemic policy is that of the rule: "To accept everything which, on the available evidence, has probability greater than 0.999, but only this," is going to have a *non-standard* world-picture—indeed an inconsistent one—but not an *impossible* one. He may well accept both *p* and *q* while rejecting their conjunction, *p* & *q*. (For him the distinction between distributive and collective conjunction becomes very real and telling: he is going to have to be very careful in "reasoning from accepted premisses"—the issue of their co-tenability (for him, *conjoint* probability in excess of 0.999) is going to be crucial.) To live in circumstances of cognitive dissonance is difficult. The human condition is difficult. But the difficult is not *eo ipso* impossible.

The risk of inconsistency is an ineliminable fact of epistemic life. Its shadow dogs every step of the quest for "a true picture of reality." Every theoretical extrapolation from the data runs the risk of clashing head-on with some other. The data themselves may conflict and cry out for theoretical reconciliation. After all, error-avoidance is not the be-all and end-all of the cognitive enterprise: "Engross truth" is no less important an injunction than "Avoid error!" And these two desiderata stand in inextricable inter-relation: the prospect of truth itself unavoidably carries with it the risk of error—and even inconsistency. There is nothing regrettable—and nothing irrational—about adopting epistemic policies that allow occasional errors—and *even inconsistencies*!—to slip through the net, provided that the general quality of the catch is high enough.

The reach of potential inconsistency is pervasive. We certainly do not need to welcome this fact. But we do have to make concessions to necessity. We strive ardently to eliminate inconsistency insofar as possible—doubtless rightly so, since it is clearly a liability rather than an asset. But there is not, and cannot be, any theoretical guarantee that this struggle will prevail and that all inconsistency can ultimately be eliminated from the arena of our cognitive struggles. But the removal of inconsistency may—in certain circumstances—exact an unacceptable price in terms of residual ignorance. The very aims and objectives of the cognitive enterprise constrain the risk of inconsistency upon us.

To be sure, this general line of approach to inconsistency is epistemological rather than ontological. But the fact remains that there is undeniably an element of plausibility in the view that reality is so complex that even our most careful efforts to engross the truth about it—as best we can—will inevitably involve contradictions. Any approach in inquiry which seeks to engross truth while yet eschewing falsehood, is bound to accept the risk of encompassing contradictions if it is of sufficient power and generality. (Think, for example, of the frequently incompatible precepts of proverbial wisdom —"A stitch in time saves nine!" and "Haste makes waste!"

to take just one instance.) No doubt we cannot join Tertullian in accepting something *because* it is absurd, but in some cases it might well be desirable to accept an inconsistent theory *in spite of* the absurdity it involves.[47]

Suppose, for example, two generally trustworthy sources of information offer the following assertions:

$S_1: p, q, r$
$S_2: p, \sim q, s$

If we did not wish to treat the conflict as vitiating altogether the utility of these sources, then we might, of course, proceed by overt agreement opting for safety by accepting only their explicit overlap. We do get something from this course, viz. p. But this leaves the (quite uncontested) contentions r and s wholly up in the air. On the other hand we might proceed *conjunctively* and swallow as our set of accepted "truths," the whole of the group

$p, q, \sim q, r, s$

(and also such of their consequences as are "unproblematic"—specifically excluding $q \,\&\, \sim q$). The prospect of enlarging our cognitive terrain (in this case by including r and s) might motivate us to being prepared to swallow some admixture of inconsistency (here that in point of q, $\sim q$), provided that utterly untoward consequences can be averted—as indeed they can.[48]

On such a perspective, we accept the idea of operating not only with regions of informational *underdetermination* (a familiar circumstance) but that of information *overdetermination* as well. We recognize the prospect of encountering informational or cognitive singularities in the systematization of our knowledge, laying the groundwork for a version of catastrophe theory in this domain—a mechanism instructing us in how to cope rationally when things go badly wrong.

The overall synthesis of our knowledge (i.e., what we *think* we know) with our meta-knowledge (our knowledge about this knowledge) affords an interesting illustration of the impetus towards inconsistency. The so-called *Preface Paradox* formulated by D.C. Makinson affords a vivid view of this phenomenon:

> Consider the writer who, in the Preface to his book, concedes the occurrence of errors among his statements. Suppose that in the course of his book a writer makes a great many assertions, which we shall call $s_1, \ldots, s_n$. Given each one of these, he believes that it is true. . . . However, to say that not everything I assert in this book is true, is to say that

> at least one statement in this book is false. That is to say that at least one of $s_1, \ldots, s_n$ is false, where $s_1, \ldots, s_n$ are the statements in the book; that $(s_1 \mathbin{\&} \ldots \mathbin{\&} s_n)$ is false; that $\sim(s_1 \mathbin{\&} \ldots \mathbin{\&} s_n)$ is true. The author who writes and believes each of $s_1, \ldots, s_n$ and yet in a preface asserts and believes $\sim(s_1 \mathbin{\&} \ldots \mathbin{\&} s_n)$ is, it appears, behaving very rationally. Yet clearly he is holding logically incompatible beliefs: he believes each of $s_1, \ldots, s_n$, $\sim(s_1 \mathbin{\&} \ldots \mathbin{\&} s_n)$, which form an inconsistent set. The man is being rational though inconsistent.[49]

Let us analyse this paradox more closely. We begin with the series of statements in the text or main body of the book: $S_1, S_2, \ldots, S_n$. (For simplicity and convenience we shall suppose that there are just two of these, i.e., $n = 2$.) Now the preface maintains that not all of these are true: $\sim(S_1 \mathbin{\&} S_2)$. The resulting over-all assertion-set $\{S_1, S_2, \sim(S_1 \mathbin{\&} S_2)\}$ is clearly inconsistent. Nevertheless, there is a strong impetus to accepting the whole of this set, and the tendency of this impetus is by no means irrational.

Now the present theory of inconsistency yields a perfectly viable resolution here, for we must simply treat the text and the preface as representing distinct assertion zones. And we would accordingly block such untoward inferences as (for example):

(1)	$t(S_1)$	givens
(2)	$t(S_2)$	
(3)	$t(\sim[S_1 \mathbin{\&} S_2])$	
(4)	$t(S_1) \mathbin{\&} t(\sim[S_1 \mathbin{\&} S_2])$	from (1), (3)
(5)	$t(S_1 \mathbin{\&} \sim[S_1 \mathbin{\&} S_2])$	from (4)
(6)	$\sim t(S_2)$	from (5)

Seemingly we arrive at an outright contradiction in the conflict between (2) and (6). But the reasoning is fallacious. The argument breaks down at step (5), which illicitly combines theses across the boundaries of distinct assertion zones. The example illustrates both how "the rational man" can come to undertake in an inconsistent set of commitments and how the present theory of inconsistency makes it possible for him to live with this situation.

In this connection, Keith Lehrer has written:

> [T]he addition of such a [preface paradoxical] belief is not worth the loss measured in terms of the objective of believing *only* what is true. . . . He has purchased the certainty of having one true belief at the price of ensuring that he has one false belief. The price is even more dear. By adding the belief which renders his beliefs inconsistent he automatically foregoes the chance of optimum success in the search for truth, that is, believing truths and only truths.[50]

But this is quite unrealistic. The epistemic *ideal* calls not just for accepting only truths, but for accepting *all* truths. Error avoidance itself can prove part of that best which is the enemy of the good. In inquiry we wish not just to avoid falsity but to engross truth. The name of the epistemic game is not the attainment of some transcendent ideal but the achievement of the best achievable balance of truth over falsity. And here the practical politics of the situation may well enjoin the toleration of inconsistency upon us.

Among the seemingly paradoxical views about truth espoused by F.H. Bradley is his insistence upon the self-contradictory character of what man knows or indeed can know in the ordinary course of things—that the totality of what we are minded to maintain as "our knowledge" at this point *in medias res* does not (and indeed *cannot*) constitute a totally consistent whole. Bradley would have it that we are forced to admit the inconsistency of what we vaunt as "our knowledge" when once the inevitable recognition of our epistemic inadequacy to truth itself comes to constitute—as it must—a part of the body of acknowledged knowledge. If truth is regarded from an epistemic light as *human* truth—what epistemically imperfect men are prepared to claim (and indeed have rational justification for claiming) to be true—then the domain of such truth will be actually inconsistent, since it cannot but include a recognition of the imperfections of our information, i.e. the knowledge that "our knowledge" contains falsehoods. The domain of our "knowledge," as best we can claim to have epistemic control over it, is outright inconsistent. Herein lies the vindication of a Bradleian insistence on the failure of our commonplace knowledge to achieve that mutual consistency which—in the traditional view of things—sets reality apart from appearance.

Consider how the "natural" endeavor to achieve the best possible systematization of our knowledge might realistically come to issue in inconsistency. Take the example of a theoretical regimentation of limited data as in the curve-fitting situation of Figure 1. Projecting a "best-fit curve basis of data obtained in region (1) alone we might well arrive at the line A. Again, on the basis of region (2) data we might well arrive at the line B. Moreover—so let us assume—no practicable, physically accessible way is open to us for securing data outside of (1) and (2). Now if our "science" were to contain two separate branches, one of which deals with the data of region (1) alone, and the other with those of (2), then an inconsistency would at once result. The prospect of a transitional phase X would not arise to yield a unified picture. The one branch of science would hold the

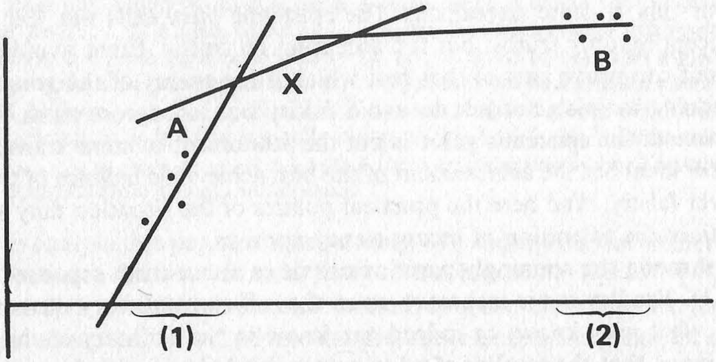

Figure 1 Curve-Fitting in Restricted Purview Cases

overall situation to be A-like, while the other would hold it to be B-like.[51]

The examples also illustrate the important point that it is not the given "facts" of the case in and by themselves, *but the very drive to achieve their smooth systematization* that produces the inconsistency at issue in such circumstances. In each case we have several *partial* perspectives on an over-all set of facts—incomplete and to all appearances mutually inconsequent. And the isolated (as it were) *systematization* of each context then leads to an extrapolation-result which is incompatible with an analogous result of the other.[52]

In this context, we must not fail to note that the compartmentalization is, after all, a basic aspect of the division of labor achieved by dividing science into branches—a part of the very reason for being of scientific specialization. (Such a compartmentalization is presumably an ineliminable feature of the structure of the scientific enterprise as we humans are able to pursue it.) And in these circumstances it is not only possible, but even likely that the resulting theories arrived at by extrapolation from an incomplete basis may prove incompatible with one another.

Indeed, a situation of this sort seems currently to be developing in natural science, as Eugene P. Wigner (Nobel laureate in physics for 1960) has detailed in the following terms:

> We now have, in physics, two theories of great power and interest: the theory of quantum phenomena and the theory of relativity. These two theories have their roots in mutually exclusive groups of phenomena. Relativity theory applies to macroscopic bodies, such as stars. The event of coincidence, that is in ultimate analysis of collision, is the primitive event in the theory of relativity and defines a point in space-time, or at least would define a point if the colliding particles were infinitely small.

> Quantum theory has its roots in the microscopic world and, from its point of view, the event of coincidence or of collision, even if it takes place between particles of no spatial extent, is not primitive and not at all sharply isolated in space-time. The two theories operate with different mathematical concepts—the four dimensional Riemann space and the infinite dimensional Hilbert space, respectively. So far, the two theories could not be united, that is, no mathematical formulation exists to which both of these theories are approximations. All physicists believe that a union of the two theories is inherently possible and that we shall find it. Nevertheless, it is possible also to imagine that no union of the two theories can be found.[53]

It is a perfectly real prospect that science might in fact evolve (in a seemingly settled way) into such a Wigner-condition of internal inconsistency. The inevitably of incompleteness and of compartmentalization assures that inconsistency can conceivably prove a real prospect—and indeed one that need not prove to be a merely transient feature of "the presently imperfect state" of the current state of our knowledge, but might well be ultimately ineradicable, affecting every realizable state thereof. One certainly cannot rule out this prospect on any grounds of general principle.[54]

To be sure, we are never absolutely *forced* to accept this sort of inconsistency as irrevocably final and as demanding an ultimate and inescapable sacrifice of the regulative principle at issue. Even if the prospect mooted by Wigner were realized—even if nature as best we grasp it is inconsistent to all scientific intents and purposes—still, this would not finally and irrevocably refute the principle of the coherence of nature. We could always tell ourselves in a hopeful tone—"If only we knew a bit more, if only we could push inquiry around the next corner, then we could eliminate the inconsistency which now confronts us; if only our information were enhanced and our science more synoptic, the difficulty would presumably be overcome."

But this line of approach is scarcely satisfactory. For although in theory we *can* always save the ideal of consistency in this way, the crucial fact remains that beyond a certain point it would—in practice—become *unreasonable*, nay Quixotic, to do so.[55]

It may even become reasonable in certain circumstances for a person to accept a set S of statements of whose inconsistency he is certain, for example when the following conditions obtain:

1. There is powerful reason for accepting each and every member of the propositional set S.
2. The set S is inconsistent (and is recognized as such).
3. Although consistency can always in theory be restored regarding S by *deleting* certain of its elements, this can (as ever) be done

in various ways, and *given the limitations of information-access and processing under which we actually labor in practice* there simply is no feasible way of justifying any one of these consistency-restoring resolutions *vis-à-vis* its alternatives.

In circumstances of this sort it would be quite reasonable to retain one's commitments to *S*—at any rate provisionally, until further notice. For in such a case, the desideratum of consistency-elimination conflicts with other cognitive desiderata (viz., adhesion to the probative standards that endorse the *S*-elements) in such a way that the latter could well outweigh the former in the specific circumstances at issue.[56]

The preservation of consistency is, to be sure, one of the prime tasks of the systematizing enterprise. And here we must look to "the other side of the coin" of the story of the blind men and the elephant. For often it is "experience" that presents inconsistencies and "theory" that *restores* harmony rather than destroying it. (Think of the old sceptics' example of sight telling us the stick is bent while touch informs us it is straight.)

It would be a mistake to think that inconsistency necessarily poses an altogether intolerable threat to the intelligibility of a cognitive enterprise. For one thing, it will be reasonable for someone to accept each statement in an inconsistent set when it is reasonable (even if presumably incorrect) for him to think this set of statements to be consistent. (As we know from the work of Kurt Gödel, there is no way in which the consistency of a set of—sufficiently complicated—propositions can ever be determined by routine, automatic means.)

Two quite different sorts of epistemic policies are available:

(1) Rather than risk contradiction within your overall set of acceptance commitments, accept nothing at all.
(2) Accept the liability of contradiction if the overall balance of epistemic benefit (i.e., margin of truth over error) is sufficiently large.

It is traditional to take the first stance. But it is also possible—and potentially advantageous—to take the second.

As William James emphasized so forcibly, the aim of the cognitive enterprise is not just to avoid error but to engross truth. To secure truths we must *accept* something: Nothing ventured, nothing gained! And to accept something rationally, we must have rules or standards of acceptance. But if these rules or standards indicate the acceptability of mutually discordant theses (as they indeed can), then there is something unsatisfying—something too pristine, purist, and pernickety—about rejecting them *en bloc* simply and solely on this account. To be sure, no sensible person would court inconsistency for its own sake. But this is not the issue. The point is that one can reasonably be in a position of tolerating inconsistencies when driven to it by the operation of (otherwise defensible) acceptance-principles. It is a crucial fact that it is the very drive towards *completeness*—itself a key parameter of systematic adequacy—that can and does so operate as to enjoin the toleration of inconsistency upon us.

But is inconsistency something we can ever tolerate in the framework of rational inquiry? Can a system ever admit contradictions without yielding up all its claims to consideration? Indeed, is inconsistency like all the other, seemingly more flexible parameters of systematicity also to be viewed as itself matter of degree?

It seems plausible to say that a system is either consistent or not—"a little bit inconsistent" seems as odd as that proverbial paradox, "a little bit pregnant." But this is misguided. The thesis that any inconsistency—no matter how minor and peripheral it may seem to be—inevitably metastasizes to spread pervasively throughout the entire system in which it figures only holds with respect to one special and highly particularized body of logical machinery.

There is no decisive *logical* (i.e. purely theoretical) impediment to the contemplation of systems purporting to characterize an inherently inconsistent nature. A steadily growing sector of recent logical theorizing has so evolved as to indicate that the automatic diffusion of contradiction is not true in general, but only in the setting of the particular framework of the logical machinery now generally characterized as "classical." Over this past generation, logicians are increasingly drawn to the view that one may distinguish between pervasive inconsistency (of the disastrous, "anything goes" form) and merely local anomalies, isolable incompatibilities whose logical perplexity is confined to within a small, localized region of a wider system.

This local-anomaly theory indicates that consistency too is a matter of degree.[57] Like the other parameters of systematicity (coherence, simplicity, and the rest), consistency is not a matter of yes-or-no but one of more-or-less. It too emerges as *a degree-admitting desideratum rather than an absolute requirement*. Like the other facts of systematicity, consistency may defensibly be sacrificed on a limited basis, in return for sufficient gains within the overall framework of systematic desiderata. Thus toleration of inconsistency is not a holus-bolus abandonment of the systematic ideal.

A consideration of the disputes initiated among quantum theorists in the 1920s and 1930s,[58] involving such issues as the fact that light quanta can be looked upon *both* as particles *and* as waves, should pave the way towards a frame of mind that grants at least the *possibility*—the *theoretical* prospect—that such a state of affairs might conceivably prove *final*. There is no guarantee that science—as far and as best as we humans can cultivate it—must issue in an account of nature and its workings from which all elements of inconsistency have been excluded. No deity has made an epistemic covenant with us which assures that our science—as it develops in actual practice—must issue in a consistent world-picture.

This general approach makes way for what might be characterized as "the synthetic impetus" whose *modus operandi* is the unification of perspectives, the fusion of aspects, compilation of diversities, etc. —an approach which rejects what from its standpoint it would characterize as "the dogma of monolithic truth," which promulgates such theses as: "Every yes-or-no question has only but one correct answer," "Among mutually confliction—albeit exhaustive—alternatives, truth will favor one unique candidate," "Differences of incompatibility are never ultimate and irreconcilable." The synthetic impetus rejects this idea that between clear alternatives the question of truth must be a matter of a one-sided monopoly.

To say all this is not to say that there are not issues where "you can't have it both ways"—clearly there are such, and lots of them. But there will—or may—also be some issues whose resolution runs along the lines of the "oracular" model of incompatible assertions conjointly intended—discordant yet equally true or valid interpretations of the same text. (Compare the Frankel passage from our collection of mottos.)

The idea that one unique alternative must prevail embodies what might be called *the myth of the God's eye view*—that all such discordancies are ultimately unreal appearances which only appertain to us beings of limited cognitive capacities. They can arise on the epistemic side but cannot possibly characterize reality. Different and discordant perspectives arise for imperfect knowers, but a perfect knower would resolve these into one unique—and uniquely correct—alternative.

Our present theory has a tendency to resist such an epistemologizing of inconsistency. It views the domain of the real as too complex and diversified to be trapped inevitably in the call of one single alternative of man's taxonomic devision. Presented with the ultimatum *A* or not-*A*, we may sometimes respond—and respond appropriately—with a willingness to take them both in stride.

The "double-truth" theory that figures prominently in discussions of medieval Averroism affords a good illustration of this stance. The theory is said to envisage an overall manifold of discordant truth, within which a proposition can be true in one domain (national philosophy) but yet contradict one that is true in another (theology), and conversely. This is one of those philosophically intriguing theories which—like solipsism—has lacked for flesh and blood defenders. (As Etienne Gilson and other scholars have convincingly maintained, no identifiable scholastic writer explicitly held so radical a position.) But while the attractiveness of the position may be limited, it is, on our present telling, not deserving of the derision usually accorded to it by historians of philosophy.

In particular, one faces the intriguing prospect that philosophy itself may be of this nature, and that the traditionally deplored absence of consensus on the key issues after two and a half millenia of serious inquiry is not due to the incompetence of the practitioners, the illegitimacy of the subject, or the meaninglessness of its questions, but to the inherently over-complex (i,e., inconsistent) character of the materials of the subject. We are thus led to contemplate a meta-philosophical view according to which the major issues of the field just do not admit of a unilateral (one-sided, consistent) conclusion. It becomes worth entertaining the idea that the philosophical interpretation of "human experience of the world"—like the interpretation of a complex text—cannot be resolved one-sidedly in favor of a particular alternative. Whatever one's sympathy towards this view—or lack thereof—one cannot but recognize that it is a perspective which the present theory does actually serve to make available.

SECTION 12

The Recent Period of Inconsistency–Toleration

The tolerance of inconsistency is certainly no eccentric innovation. From the days of Zeno and Parmenides in antiquity to those of Kant, Hegel, and the neo-Hegelians of modern times we have been exposed by able theoreticians to the idea of "paralogisms," "antinomies," and "paradoxes"—arguments in logic, mathematics, and natural philosophy that build up a cogent case on both sides of a significant question. It does occasionally seem as though the best picture we can form of some aspect of "reality" depicts it as a superposition of incompatible states of affairs.[59] One possible way of reading some of the ancient sceptics is not so much as maintaining that rational inquiry cannot build up *any* picture of the world, but rather that it cannot build up any *consistent* picture of the world, in that knowledge is not unattainable *as such*, but unattainable under the conditions—especially the consistency conditions—generally imposed upon it.[60] The Hegelian tradition in particular has become the channel in recent philosophy of the old view that reality is so complex and changeable that any attempt to construct a comprehensive account of it must involve inconsistencies. Here too it is germane to recall the vision of some post-Hegelian philosophers (e.g., Dilthey), who regard philosophy as *Weltanschauungslehre*, and as unfolding a vast metasystematization which embraces distinct and mutually conflicting systematizations among its assertion-zones. This perspective views the domain of philosophical truth as too rich for the confines of a single self-consistent system (as per our motto from Walt Whitman).

The last decade or so has seen the stirrings of a marked reorientation in thinking about the issue of consistency. This change has been visible on several different fronts, especially the following:

Marxist Dialectics

As Soviet theoreticians have developed it in recent times, in the wake of Engels' intimations,[61] it is the characteristic feature of dialectical theory to espouse "the dogmatic assertion that there are real contradictions in the world"[62]—that, any given stage of world history, contradictions can hold in that certain theses and their negations will both be true together. In the wake of these ramifications in Marxist theory, various recent Soviet writers have endeavored to devise a "dialectical logic"—which is not, however, a "logic" in the sense of modern logical theory, but rather in the older, 19th century, metaphysical mode that took its inspiration from Hegel.[63] The idea is to clothe the older framework of a Marx-Engels Hegelianism with trappings borrowed from various aspects of modern science. The utility and promise of such metaphysical theorizing is a matter of dispute—even in the Soviet Union itself. (In the West it has usually been dismissed out of hand.[64]) Recently, however, various formal logicians—particularly in eastern Europe—have tried to articulate these theories in the rigorous format of a formal calculus. This gives the matter a rather different aspect, to which we shall return below.

Formal Reconstructions of the Hegelian Dialectic

Over the past decade there have been various efforts to provide a "rational reconstruction" of the formal apparatus of Hegel's dialectic, using the mechanisms of symbolic logic.[65] This new venture in Hegel-interpretation—in the formalization of logical ideas of readings (or misreadings) of Hegel—may or may not eventually yield results which succeed on their proper ground of providing the basis for a new and firmer basis for understanding Hegel's philosophical ideas. However, quite independently of this, they have a substantial interest in their own right, as producing logical formalisms of novel types. In any case, however, they have provided a powerful stimulus and support to the expanding project of developing formal logics of inconsistency to which we now turn.

"Dialectical" Symbolic Logics

Recent years have seen a diversified spate of attempts to construct formal systems of symbolic logic that are capable of admitting inconsistencies without vitiating consequences.[66] A forerunner he was a pioneering 1948 paper by the Polish logician Stanislaw Jaskowski,[67] but the main thrust of effort is of more recent date. A great deal of work has been done since the early 1960s by the Brazilian logician Newton C.A. Da Costa and his associates,[68] who have tried to devise suitable systems of inconsistency-tolerating "dialectical" logic, intending to provide for an inconsistent set theory in which Russell's paradox is derivable, but which nevertheless avoids hyperinconsistency. Other contributions in this general direction of dialectical logic include papers by F.G. Asenjo,[69] S.K. Thomason,[70] Richard and Valerie Routley,[71] Robert K. Meyer,[72] Graham Priest,[73] and others. (This literature is only now beginning to reach a stage of rapid growth—at this writing, much of it is accessible only in preprint versions.) The relevant logic of A.R. Anderson and N.D. Belnap, Jr. and their associates should also be mentioned under this head.[74]

A striking conclusion emerges when our present theory of inconsistent possible worlds is viewed against the background of these dialectical logics. Despite their shared concern for making the acceptance of inconsistencies a rationally viable option, these two lines of approach are entirely disjoint from one another. For *the present approach*—as we have seen—*dispenses entirely with any need to modify the principles of classical logic.* Despite its provisions of a non-standard ontology and a non-standard semantics, nevertheless, at the crucial level of logical machinery, it requires no innovations or renovations whatsoever.

Unorthodox Quantum Physics

In recent physical theory the idea of an inconsistent world has come into its own in being advanced as a serious scientific theory, moving along the roadways of the physical literature under the name of the Everett-Wheeler theory in quantum mechanics.[75]

The focal point of this theory is the issue of measurement in quantum theory, specifically the well-known "problem of the reduction of the wave packet." With such quantum-theoretic measurement cases as the nucleonic decay-timespan of a very heavy radio-active element, the result of a measurement is formally speaking a superposition of vectors, each representing the quantity being measured as having *one* of its possible values, and each thus providing a distinct observational result of measurement. The obvious difficulty is how this *superposition* of distinct outcomes can be reconciled with the fact that in practice only one value is ever to be observed. How can a measurement force reality to make up its mind, so to speak, between physically parallel alternatives? How can the process of actual observational measurement of itself constrain an inherently pluralistic situation into producing a unique result?

The orthodox quantum-theoretical line of response to this question is to say that only one outcome is in fact real, and that the other alternatives are *unactualized possibilities*, merely possible but utterly non-actual alternatives. This approach, however, at once encounters a serious difficulty: How can an experimental trial of physical measurement single out as uniquely real and actual one specific situation whose status in all departments of physical theory is altogether similar to that of others? If the state vector is construed as presenting alternative *possibilities*, what in the name of heaven or, rather, of *physics*) reduces these to a single *actuality*? Given that physics is inherently non-discriminatory as between these alternatives, how can we rationalize the fact that the measuring process is able to select one single alternative as the uniquely real observed value? How is it that in experimental trials at quantum measurement only one unique single outcome can be encountered observationally, when the theory itself provides no means for collapsing the state vector at issue into merely a single one of its values?

The Everett-Wheeler hypothesis cuts the Gordian knot of this problem by the daring tactic of insisting that *all* of the possible alternative outcomes are in fact actual. We come here to its notorious hypothesis of the "self-multiplication of the universe." Intuitively, its physical picture is that of a universe continually splitting into a multiplicity of distinct but equally real subworlds, each embodying a unique but definite result of the quantum measurement. The cosmos is the internally complex counterpart of a linear superposition of vectors, each of which represents observable reality as having assumed one of its value outcomes. The seeming uniqueness of the quantum observation is a simply *perspectival* aspect of the relationship between the observer and what is observed: being placed within the subworld where a given result obtains, the other no less real outcomes are simply observationally inaccessible to the observer. The reason why all observers agree on a given result inheres in the merely parochial fact that they hail from the same subworld, and accordingly lack all prospect of causal interaction with the rest. We have simply lost to another subworld those observers whose view of reality conflicts with our own.

It is worth noting that this doctrine carries to its logical conclusion the tendency of the Copernican revolution to move away from the anthropocentrism of the Aristotelian world-picture. We standardly draw the familiar distinction between this, the actual world and other physically possible but unrealized worlds. From the standpoint of the Everett-Wheeler hypothesis it is unduly anthropocentric to view this world of ours as the uniquely actual one: We are bid to recognize the other physically realizable worlds as wholly on a par with ours in point of actuality. That they are inaccessible to us is—as it were—our misfortune, and not theirs. The world is *the stage of a concurrent realization of incomparable alternatives*. Only the parochial limitations of our limited sensory perspective presents us men from perceiving this fact. Where other approaches see *incompatible alternatives*, the Everett-Wheeler hypothesis sees *concurrent actualities*. But since these are, in the very logic of the situation, mutually exclusive, the systematization of quantum-physics by the Everett-Wheeler approach invites (though it does not irrevocably demand)[76] a logical apparatus that is inconsistency-tolerant.

And even quite apart from the Everett-Wheeler theory, who can say with confident assurance that the next twist and turn of elementary particle physics will not thrust us into the world envisaged by an inconsistency-tolerant logic of world-superposition. It is surely not beyond the pale of conceivability that the smoothest theoretical systematization in subatomic physics should call for the supposition that this envisages actual inconsistencies in nature.

The Meinong Revival

The 1970s have seen a rapid rise in the stock of Meinong's *Gegenstandstheorie* on the market-place of philosophy. In particular, several logicians have sought to rehabilitate Meinong's theory of "impossible" objects.[77] It is no exaggeration to say that a small but powerful "back to Meinong" movement is astir, a movement which lends yet another dimension of support to the demise of the traditional flight from inconsistency.

* * *

The foregoing survey is highly suggestive from our present perspective. For it indicates that a rather striking change of attitude is in the air as concerns contradictions and inconsistencies—alike among logicians, scientists, and philosophers. While these various approaches unquestionably differ radically from that of our present theory, they serve to place this theory within the framework of a larger setting whose recognition conduces to a just appreciation of our present concerns.

The project of inconsistency-admitting formal systems is a fair way to becoming a significant area of current research, one which, on present indications, may well turn out to constitute a major stage in the development of logical theory in the second half of the 20th century. Logicians have looked inconsistency in the eye and are not totally horrified by what they see. Many of them no longer believe that rationality will necessarily fly out the window when inconsistency comes in the door. The present essay should be understood against this background. To be sure, its own approach to inconsistency is quite different from those just surveyed. But differences of approach aside, it does share with them the important ideological feature of inconsistency-toleration.

Throughout history, a sort of *horror contradictionis* has been endemic among rigorous thinkers, and it has been the view of the logical guild that once a contradiction has been encountered nothing more remains to be done but to leave the scene with proper expressions of disapproval. The indications are that a new spirit is abroad nowadays. Logicians and philosophers are coming to take a new and more tolerant view of inconsistency.

The Wittgensteinean Démarche

One of the key leitmotifs of Wittgenstein's *Foundations of Mathematics* (Oxford, 1956) is an attack on setting contradiction-avoidance up as a be-all and end-all. ("Der Widerspruch. Warum grad dieses *eine* Gespenst. Das ist doch sehr verdächtig." III.56.) We can *decide* to exclude contradiction as inconsistent with our purposes in mathematicizing, but the boundary we erect around it is not an insuperable one. ("Der Zaun, den ich um den Widerspruch ziehe, ist kein Über-Zaun," III.87.) In fact, moreover, its is simply wrong to think that contradiction destroys the whole effort—this is a mere myth that we can shatter if only we are imaginative enough (V. 12). It is wrong to think that mathematics must be fitted out with a contradiction-free foundation (V.13 and V.21). Contradictions need work no irreparable harm. We can deal with contradictions when and where we come to them—but there need be nothing amiss with the earlier stages of an argument that ultimately leads to an abyss someplace (II.78 and II.81). Indeed if a contradiction were actually found in arithmetic, this would only show that an arithmetic with *such* a contradiction could render very good service; and it would be better for us to revise our concept of the certainty required here than to say that what is at issue is really not a genuine arithmetic (V. 28). No discovery of an obscure inconsistency in the "foundations" of arithmetic could show that our calculations were misconceived; no such discovery could nullify the fact that what we are and have been doing here is real *calculation* (V.12). We may need luck or acumen (or both) to avoid difficulty in mathematics, but that just makes mathematics akin to the rest of life (V.13)—mathematics is, after all, an anthropological phenomenon (V.26). As Wittgenstein explicitly insists: "Mein Ziel ist, die Einstellung Zum Widerspruch und zum Beweis der Wiederspruchsfreitheit zu ändern." (II.82) His declared aim is to overcome what he condemns as: "Die abergläubliche Angst und Verehrung der Mathematiker vor dem Wiederspruch." (I.20; cf. also our motto passage.)

SECTION 13

Ontological Import

The ensuing portions of this essay will examine various reasons for regarding non-standard possible worlds and the objects they comprise as ontologically on a par with standard, consistent and complete worlds and their objects. Our focus has been on the tolerance of inconsistency within a cognitive system. The account offered of such systems facilitates the countenancing of an inconsistency which is not merely *epistemological*—a matter of inconsistent beliefs in some cognitive system—but *ontological*, in envisioning an inconsistent subject-matter for such a system. Our account as presented so far envisages non-standard possible *worlds* which are constructed out of other worlds by schematization and superposition, rather than just sets of sentences which are the unions and intersections of sets of sentences satisfied by standard possible worlds. These non-standard worlds are strange entities, themselves inconsistent and incomplete. While inconsistent and incomplete *beliefs*, or sets of sentences, are fairly familiar, inconsistent and incomplete *objects* or facts are another matter entirely. Looked at from this point of view, our ontological suggestions are likely to seem not only objectionable on metaphysical grounds, but otiose. For why should one think of our apparatus for dealing with inconsistency as involving the construction of new kinds of *worlds* at all? Why should one not accept that construction as having provided a new and useful way of talking about sets of *sentences* which happen to be inconsistent or incomplete? One could continue to think of those sentences as the intersections and unions of sets of sentences which are true in standard consistent and complete worlds, but why should one postulate also a non-standard, inconsistent or incomplete *world* in which all and only those sentences are *true*? Put another way, we have so far provided a semantics for those special sets of sentences which is developed in terms of the standard semantics for possible worlds. But semantics is not yet ontology.

In fact, the semantics outlined above is parasitic on the standard semantics, not parallel to it. A semantics for a theory expressed in some language should comprise three different kinds of account. First, one wants a specification of the semantic correlates (usually denotations) of the primitive terms and predicates. This may be given in the form of a set of models for the theory, or in the form of some recipe for determining the semantic correlate of an expression on the basis of its use (as for instance in causal-historical theories of reference). Next, one seeks a recursive compositional theory to stipulate how the semantic correlates of compound or complex expressions are built up functionally out of the semantic correlates of component expressions. Finally, a description of what complete expressions of the language (sentences) are asserted (as true) by the particular theory in question is needed. The semantic account which we have offered of incomplete or inconsistent theories in terms of non-standard possible worlds performs all of these essential functions of a semantic theory. But it may be noticed that it does not offer specifications of *any* of the three fundamental parts of semantic theory which are *independent* of the semantic account of consistent and complete theories in terms of standard possible worlds. The semantic correlates of all basic expressions are taken to be just the same as for ordinary languages. And compositional principles which allow us to interpret compound expressions are also taken over wholesale from the standard semantic account. Only the specification of the sentences asserted by the theory whose semantics is being constructed differs from the standard account. Yet even this specification is offered in terms of the intersection and union of consistent and complete theories semantically interpreted in the ordinary way. Now all of this dependence on the ordinary semantics *might* be taken as indicating the judicious conservatism of the extension of standard semantics which we are proposing. But it might also be taken as indicating the ontological superfluity of non-standard possible worlds for the semantic project of interpreting inconsistent and incomplete theories. Why, it might be asked, should non-standard possible worlds not be treated as derivative, second-class entities which can be reduced to or eliminated in favor of standard possible worlds in the interests of ontological parsimony and precision? (Since we can specify and interpret semantically the class of claims made by inconsistent or incomplete theories in terms of the union and intersection of sentences true in those standard worlds.) In short, isn't the talk of non-standard possible worlds just a façon de parler to which no ontological weight should be attributed?

We will argue that such easy dismissals of the ontological claim of the non-standard semantics for inconsistent and incomplete theories are not warranted, and that in fact non-standard possible worlds are completely on a par ontologically with ordinary possible worlds. We shall claim that (the polemic proffered above notwithstanding) we can with equal justice consider consistent and complete possible worlds as verbal inventions, to be construed ontologically as complex constructions out of the ontologically basic non-standard worlds. In particular, we shall examine three sorts of considerations which may be advanced in defense of the value of an ontological commitment to standard possible worlds, in order to show that, properly developed, those considerations apply equally in favor of such a commitment to non-standard possible worlds. We are *not* concerned to argue that possible worlds should be considered as part of the furniture of the universe, only that there is nothing to choose between standard and non-standard possible worlds in this regard. This will be called *the parity thesis*. Let us consider some circumstances and projects for which standard possible worlds have been useful.

The notion of possible worlds has been available since Leibniz, but it was not until Kripke showed how to use accessibility relations defined on such worlds to provide a semantics for modal logic that there seemed to be any reason to expand our ontology to include such entities.[78] Carnap had shown how state descriptions, maximal consistent sets of sentences, could be exploited formally to good advantage when doing semantics.[79] But there was no urge to interpret such state descriptions as *worlds* as long as the explanations at issue utilized no relations between them over and above the relations between the sentences they contained. Such state descriptions were just constructions out of sentences, and did not need to be regarded as involving any commitment to the existence of anything except sentences and sets of sentences. Kripke showed how to *explain* the syntactic relations between sets of sentences captured in the various modal logics which Lewis and Langford had botanized years before;[80] by appealing to accessibility relations between the *worlds* in which those sentences are true. His completeness and soundness results, demonstrating the equilibrium obtaining between the syntactically specified modal logics and the semantically valid claims generated by the various kinds of accessibility relations, gave possible worlds a distinct explanatory role in a theory of truth and meaning for modal discourse. Precisely because such a theory had proved elusive as long as people thought in terms of sets of sentences rather than non-linguistic entities like possible worlds by means of which modal statements might be interpreted, possible worlds acquired a strong claim to ontological consideration.

In the next section of this essay we will argue that whatever reasons the explanation and interpretation of modal discourse by a Kripke semantics give us to commit ourselves to the existence of standard, consistent and complete possible worlds, the same reasons apply equally to non-standard, inconsistent or incomplete possible worlds. We will present a complete Kripke semantics for locally inconsistent or incomplete modal discourse, in terms of accessibility relations between non-standard possible worlds and the usual Kripke definitions. Recall that we do not want to argue that we should treat *any* possible worlds as real, only that the considerations which can be advanced in favor of so treating standard possible worlds apply equally to non-standard possible worlds.

It is not, however, to modal discourse that contemporary analysis looks for enlightenment about ontology, but, as Quine has taught, to quantificational discourse. Indeed, it is only because the interpretation of propositional modal discourse according to Kripke's method requires quantification over possible worlds in the metatheory (to make essential use of expressions like "truth in *all* possible worlds accessible from a given world") that his account offers such strong prima facie reason to add possible worlds to our ontology. Having shown how to extend our semantics on non-standard possible worlds to interpret modal discourse we will show how to extend that apparatus to interpret quantificational discourse. Here the concern will not be with quantification over inconsistent and incomplete *worlds*, but over inconsistent or incomplete *individuals* inhabiting those worlds.

The ultimate aim of our construction is to make it possible to take seriously a Meinongian ontology of such individuals. In line with Quine's famous dictum "To be is to be the value of a variable", the expression of his general view that a theory is ontologically committed to the existence of the elements of whatever domain the quantifiers employed in that theory are taken as ranging over, the detailed development of such an ontology requires us to consider discourse involving quantifiers ranging over inconsistent or incomplete individuals. We accordingly show how to develop our semantics so as to accommodate quantified statements which are to be interpreted by means of non-standard worlds. The argument will then be that if inconsistent and incomplete *individuals* exist, then *a fortiori* so do inconsistent and incomplete worlds.[81]

Another dictum of Quine's concerning the ethics of ontologizing provides the occasion for a further development of the Meinongian universe. "No entity without identity", requires us, plausibly enough, to be able to individuate, at least in principle, the entities we quantify over in order for that quantification to count as legitimate for the purposes of determining ontological commitments. Quine has used this slogan effectively in polemics against establishing such commitments to the existence of possibilia in general, as well as to the existence of intentional entities. The individuation of inconsistent and incomplete individuals is clearly a problem which a Meinongian ontology must face. Basically, that problem can be divided into two parts. First, for inconsistent individuals, how are we to keep the inconsistencies localized or quarantined so as to be able to say that there are different such individuals, rather than just one of whom everything is true? Second, for incomplete individuals, how are we to treat incomplete individuals, one of which *includes* the other, for the purposes of our identity relation? That is, given an incomplete individual with exactly properties $P_1, \ldots, P_n$ and an individual with exactly properties $P_1, \ldots, P_{n-1}$, are we to say that they are identical, or not? And what is their relation to an incomplete individual which has only the property P_n true of it? Can any coherent theory of such individuals be developed? In terms of the detailed model-theoretic account of non-standard possible worlds, a theory of identity and individuation for such inconsistent and incomplete individuals will be developed, following formally our earlier treatment of the analogous problems for inconsistent and incomplete *worlds*.

Our general strategy up to this point will have been to exhibit non-standard individuals, their properties and relations, and the non-standard worlds they inhabit as constructions out of standard individuals, relations, and worlds, thereby producing a consistent and complete metatheory accounting for inconsistent and incomplete entities. It ought not to be concluded from this strategy that non-standard worlds are ontologically derivative, *mere* constructions out of the ontologically more basic standard entities, however. For as we shall show in the concluding sections of this essay, it is equally possible to treat non-standard worlds and their inhabitants as basic, reducing standard worlds and their inhabitants to ideal constructions out of those non-standard entities. Indeed, we shall argue that the process invoked to demonstrate *parity of ontoligical status* between standard and non-standard worlds and individuals corresponds to a human activity of the greatest possible philosophical interest, namely empirical cognitive inquiry. We shall also claim that crucial formal characteristics of such inquiries, e.g., the convergence to a limit which is a *sine qua non* of cognitive success in such an enterprise, can be fully illuminated only by taking into account the relationship of standard to non-standard entities which establishes just this ontological parity. This discussion of empirical methodologies will serve, finally, to indicate issues for which the postulation of non-standard entities is of paramount philosophical interest and usefulness. So much, then, for a preview of what lies ahead.

SECTION 14

Modal Logic on Non-Standard Possible Worlds

The prime explanatory function which possible worlds perform (and certainly the function which was responsible for their current popularity as a formal logical device) is that of interpreting modal discourse. Although we have not explicitly dealt with modal languages, to the extent to which our non-standard possible worlds are properly called "possible worlds" we would expect it to be possible to interpret modal discourse in terms of them. Indeed, our method for constructing non-standard possible worlds by superposition and schematization extends straightforwardly to the case in which the base or generating standard consistent and complete worlds contain modal claims as well. A modal claim, just like a non-modal one, will hold in a schematized world just in case it holds in both or all the generating worlds, and will hold in a superposed world just in case it holds in one of the generating worlds:

(i) $/\Diamond p/_{w_1 \dot{\cup} w_2} = T$ iff $/\Diamond p/_{w_1} = T$ or $/\Diamond p/_{w_2} = T$
(ii) $/\Box p/_{w_1 \dot{\cup} w_2} = T$ iff $/\Box p/_{w_1} = T$ or $/\Box p/_{w_2} = T$
(iii) $/\Diamond p/_{w_1 \dot{\cap} w_2} = T$ iff $/\Diamond p/_{w_1} = T$ and $/\Diamond p/_{w_2} = T$
(iv) $/\Box p/_{w_1 \dot{\cap} w_2} = T$ iff $/\Box p/_{w_1} x = T$ and $/\Box p/_{w_2} = T$.

Just as in the non-modal case, some of the superposed worlds will be inconsistent. All of the sorts of non-modal inconsistencies will still be possible in these worlds, as well as some new ones added by the special modal vocabulary. Thus we may have:

$$/\Box p/_{w_1 \dot{\cup} w_2} = T \text{ and } /\sim\Diamond p/_{w_1 \dot{\cup} w_2} = T.$$

For it may be that

$$/\Box p/_{w_1} = T \text{ and } /\sim\Diamond p/_{w_2} = T.$$

But we will not have

$$/\Box p \mathbin{\&} \sim\Diamond p/_{w_1 \dot{\cup} w_2} = T$$

because as long as we start out with standard, consistent and complete worlds w_1 and w_2 it will never be the case that

$$/\Box p \mathbin{\&} \sim\Diamond p/_{w_1} = T \text{ or } /\Box p \mathbin{\&} \sim\Diamond p/_{w_2} = T.$$

It is clear that for the modal case, just as for the non-modal case, all the logically true statements and one-premise inferences which hold in standard worlds also hold in *all* of the non-standard possible worlds constructed out of standard ones by superposition and schematization. If the sentences made true by all of the standard worlds we consider are governed by the axioms of the modal system **S5**, then those axioms will govern also the sentences true in the non-standard worlds constructed out of them, and similarly for all the other possible modal logics. Of course, as before we must read all of our inferential rules collectively, rather than distributively. But in no respect do modal statements differ from non-modal ones in the ways in which they must be handled in superposed or schematized worlds.

But to have said this much is not yet to have offered a semantics for modal logic in terms of non-standard possible worlds in the same sense in which Kripke originally presented a semantics for modal logics in terms of standard possible worlds. For as things stand we can interpret modal claims on non-standard possible worlds only by translating them into modal claims in the standard worlds out of which we have constructed the others. And the modal claims in standard worlds admit of interpretation by means of accessibility relations between such worlds. We have, then, so far only offered a semantics for modal claims in non-standard worlds which is derivative from the Kripke semantics for modal claims in standard worlds. If this were indeed the only way to provide a semantic interpretation of modal claims in non-standard worlds, then one would be justified in concluding that such worlds should be assigned a derivative, second-class ontological status with respect to standard worlds.

It is, however, as we shall see, possible to extend the accessibility relation for standard worlds to non-standard ones in a natural way so that the valuations of modal claims in non-standard worlds can be determined directly from the accessibility relations between non-standard worlds, just as in standard worlds. It is clear that in developing such an extension we must be careful to ensure that no new worlds (or only redundant ones) become accessible from standard worlds. For if we did allow that, we would have to give up one of the cardinal virtues of the non-standard world approach, namely that it is a logically conservative extension of the domain of standard, consistent and complete worlds. All logical theses, modal and non-modal, are to hold in all non-standard worlds. Kripke's completeness results show that we cannot change the class of accessible worlds relative to any standard world without affecting that logic. So what we want is to define an accessibility relation which holds exclusively between non-standard worlds, and which makes true in the non-standard worlds (under the standard Kripke interpretation) just those modal claims endorsed by the straightforward extension of our non-standard semantics to the modal case, as indicated above. So we want to be able to define a relation R′ between non-standard possible worlds such that:

(a) $/\Box p/_{w_1} = T$ iff for all w_i such that $w_1 R' w_i$, $/p/_{w_i} = T$
(b) $/\Diamond p/_{w_1} = T$ iff there is a w_i such that $w_1 R' w_i$ and $/p/_{w_i} = T$

We want the semantics for modal claims on non-standard possible worlds given by (a) and (b) (the Kripke semantics) to coincide with the semantics for modal claims on non-standard possible worlds given by (i)–(iv) (the non-standard semantics).

Consider, then, the following definition of the extended accessibility relation R' in terms of the accessibility relation R which we suppose to be already defined for standard possible worlds:

(1) If $w_1 R w_2$, then $w_1 R' w_2$
(2) If $w_1 = w_i \frown w_j$ for some i, j, then $w_1 R' w_2$ iff there are w_k, w_m such that $w_2 = w_k \frown w_m$ and $w_i R' w_k$ and $w_j R' w_m$.
(3) If $w_1 = w_i \cup w_j$ for some i, j, then $w_1 R' w_2$ iff there are w_k, w_m such that $w_2 = w_k \cup w_m$ and $w_i R' w_k$ and $w_j R' w_m$.

This is a recursive definition of R' in terms of R, where the ordered pairs of worlds which make up R are pairs of standard worlds and those making up R' are either pairs of standard worlds or pairs of non-standard worlds. Since it is recursive, it defines R' relations also among non-standard worlds constructed by iterations of both superposition and schematization, for instance for worlds of the form $w_1 \cup w_2 \cup w_3 \cup (w_4 \frown w_5)$. A world is R' accessible from a schematized world just in case it is itself constructed by the schematization of two worlds, each of which is R' accessible from one of the worlds intersected to form the original schematized world. A world is R' accessible from a superposed world just in case it is itself constructed by the union of two worlds, each of which is accessible from one of the worlds unioned to form the original superposed world.

It is shown in Appendix II that the constructed relation R' will share the algebraic properties of the generating accessibility relation R. In particular, R' will be reflexive, or symmetric, or transitive, if R has these properties. We have already seen that the semantics of non-standard worlds is logically a conservative extension of the semantics of standard possible worlds, in the sense that whatever logic is stipulated to govern the sentences true at standard worlds will apply equally to the non-standard worlds constructed out of them, for modal claims as well as non-modal ones. Thus if the sentences satisfied by some set of standard consistent and complete worlds is governed by the modal system **S5**, for instance, that same logic will hold for all of the non-standard worlds constructed out of that basis. Kripke has shown how to relate different syntactically specified modal logics, such as **S5**, with algebraic properties of the accessibility relation (in the case of **S5**, with the reflexivity, symmetry, and transitivity of that relation) in such a way that the sentences validated by a semantics based on that relation and principles (a) and (b) above are just the sentences syntactically generated by the modal logic. We have shown so far how to construct an extended accessibility relation for non-standard worlds which will share the algebraic properties of the generating accessibility relation on standard worlds, and have shown that our semantics for the non-standard worlds will validate just the same logical principles and rules of inference (interpreted collectively rather than distributively) as are valid in the standard, generating worlds. Thus, for instance, if the accessibility relation R on the standard worlds is reflexive, symmetric, and transitive, the sentences true at the standard worlds will be governed by the logic **S5**. That same logic will govern the sentences true at non-standard worlds constructed out of those basis worlds, and the non-standard worlds will be related by an extended accessibility relation with the same algebraic properties as the original relation. What remains to be shown is that the sentences validated by our semantics on non-standard worlds are just the sentences validated by the Kripke semantics on those same worlds, using the extended accessibility relation and Kripke's principles (a) and (b). Proving this will not only extend Kripke's completeness results to modal logic interpreted on non-standard possible worlds, but will show the equivalence of the non-standard semantics to the Kripke semantics utilizing the specially extended accessibility relation. Such a proof may be found in Appendix III.

It is thus possible to define an accessibility relation which allows us to give a Kripke semantics in terms of non-standard possible worlds for modal logic which allows inconsistent and incomplete theories. Further, by proving in Appendix III that this extended Kripke semantics is equivalent to our non-standard semantics for modal claims interpreted on non-standard possible worlds, and that non-standard semantics validates just the logical principles and rules valid in standard possible worlds, we have shown that the various modal logics are *complete* with respect to the extended Kripke semantics—for instance, that exactly the theorems of **S5** will be validated by a reflexive, symmetric, and transitive accessibility relation R' defined on non-standard as well as standard worlds, just as those theorems are validated by such an accessibility relation defined only for standard possible worlds. Thus as far as modal logic is concerned, we can treat non-standard possible worlds as completely on a par with standard ones, interpreting modal claims in each on the basis of accessibility relations with other (standard or non-standard) possible worlds.

SECTION 15

Model Theory and Quantification

In several early sections of this essay we mention the inconsistent and incomplete individuals which inhabit non-standard possible worlds constructed from standard worlds by superposition and schematization. We shall now expand this treatment by offering a theory of how quantification with respect to those individuals ought to proceed, and how such individuals can be identified with and individuated from each other. Until such an account is in hand, we have not specified in full detail a model theory for the inconsistent and incomplete sets of sentences which result from intersecting and unioning the consistent and complete sets of sentences satisfied by standard worlds.

To see this, consider the following objection to our construction as we have developed it so far: while no insuperable problem arises when inconsistent and incomplete sets of sentences are constructed by union and intersection, no reason has yet been offered to take seriously the interpretation of such sets of sentences in terms of non-standard *possible worlds*. Furthermore, it can be shown that the semantics for such sets of sentences which we get by interpreting them in terms of the intersections and unions of the standard worlds which are the intended models of the consistent and complete sets of sentences we started with is not sound and complete. That is, if we take two standard possible worlds w_1, w_2 and the sets of sentences satisfied by them $T_1 = \{p : /p/_{w_1} = T\}$ $T_2 = \{p : /p/_{w_2} = T\}$, in the usual model-theoretic sense of satisfaction of a set of sentences by a structured domain—with relations on it specified as corresponding to the predicates of the language of the theories T_1, T_2,—there will be sentences in the intersected theory $T_1 \cap T_2$ that are not satisfied in the ordinary sense by the schematized world $w_1 \cap w_2$. For suppose that the domain of w_1 consists of exactly n things, and the domain of w_2 consists of exactly n things, but only two of those things are in both w_1 and w_2. Then the domain which is the intersection in the ordinary sense of w_1 and w_w will consist of exactly the two things which existed in both w_1 and w_2, but the theory $T_1 \cap T_2$ will include the statement that there are exactly n things. Since the statement "There are exactly two things" was not included either in T_1 or in T_2, it does not appear in $T_1 \cap T_2$, and yet it is satisfied by the model structure $w_1 \cap w_2$. Thus the intersection of the two worlds semantically satisfied sentences like "There are exactly two things," which are not in the set $T_1 \cap T_2$, and it fails to satisfy sentences like "There are exactly n things," (where $n \neq 2$) which do appear in the theory $T_1 \cap T_2$. Thus the semantics which is offered for theories constructed by the union and intersection of sets of sentences satisfied in standard consistent and complete worlds in terms of the union and intersection of those *worlds* is neither sound nor complete. This being the case, the objection concludes, schematized and superposed worlds don't provide a *semantics* for inconsistent and incomplete theories at all, and we have no reason whatsoever to extend our ontology to include any inconsistent or incomplete objects save sets of sentences.

With the core of this objection there can be no disagreement. If we think of standard possible worlds as algebraic structures consisting of a set of elements (the domain) and a set of tuples of domain elements (the predicates and relations pertaining to that domain) and interpret theories formulated in first-order languages according to the usual definition of satisfaction by a model, then the set of sentences satisfied by the set-theoretic intersection of two models will not in general be the intersection of the set of sentences satisfied by the first model and the set of sentences satisfied by the second. We now must ask whether granting this point is fatal to the enterprise of a Meinongian ontology of non-standard possible worlds comprising inconsistent and incomplete individuals.

The objection will be fatal to that project only if the schematized world $w_1 \cap w_2$ is interpreted as the set-theoretic intersection of worlds w_1 and w_2, and this is *not* the way the expression "$w_1 \cap w_2$" was defined when it was introduced. The definition of world conjunction or schematization was as follows:

> $w_1 \cap w_2$ is that world such that, for any proposition P, P obtains in this world iff P obtains *both* in w_1 *and* in w_2: $|P|_{w_1 \cap w_2} = T$ iff $|P|_{w_1} = T$ and $|P|_{w_2} = T$.

The argument presented above should not be taken to be an objection to the notion of non-standard worlds, but rather as a demonstration that schematized worlds—those which satisfy all and only the propositions satisfied by *both* of two other worlds—*must not* be interpreted simply as the set-theoretic intersection of those basis worlds. Construction of a schematized world proceeds by a process of ontological *fusion* of two other worlds; fusion which results in a world which satisfies just the intersection of the sets of sentences satisfied by the original worlds, but which cannot itself be identified with the set-theoretic intersection of those worlds. We marked this point in the preliminary fashion appropriate to that stage of our exposition by pointing out that if we consider a world w_1 with exactly the three individuals a, b, and c in it, and a world w_2 with exactly a, and d in it, and a relation R such that $/aRb/_{w_1} = T$ and $/aRd/_{w_2} = T$, $/\exists xaRx/_{w_1 \cap w_2} = T$ although there will be no individual we can identify as b or d which is R-related to a in the schematized world. Such schematized worlds do not, in other words, have as their domains the intersections of the domains of the worlds out of which they are constructed by schematization.

To say all this is not, however, to say that the objection raised above is pointless, it is merely to restate its point. For having granted that the sort of ontological fusion of worlds which creates schematized worlds (for example) is not set-theoretic intersection, though it satisfies a set of sentences constructed by such an operation, we must ask what sort of ontological operations schematization and superposition *are*.

So far we have talked about *the* world which satisfied all the sentences satisfied by both of two other worlds, or by either of two other worlds, but we have not tried to offer a precise characterization of these worlds which is independent of the semantic role we want them to play in interpreting inconsistent and incomplete sets of sentences. That is, we have been concerned up until now to show that there were good reasons to want a semantic interpretation of such sets of sentences, and that such an interpretation utilizing nonstandard possible worlds can be coherently described and plausibly defended. Insofar as one is interested in the ontological reality of such objects, however, one needs more than a coherent description of their role as semantic place-holders. For ontological purposes we must say more about superposed and schematized worlds themselves. We must be able to describe them independently of their role as interpreters of inconsistent and incomplete theories. In particular, we would like to be able to describe the two modes of world-fusion, schematization and superposition, without appealing to sets of sentences at all. As initially presented, the previous objection argued correctly that such an account could not be offered in straightforward terms of set-theoretic intersection and union of worlds, and concluded incorrectly that one could not be given at all. Our present task is thus to develop an alternative ontological account of world-fusion—one which will produce objects that can play the semantic role we have described in the interpretation of inconsistent and incomplete theories.

We want to define algebraic operations of schematization and superposition on standard model-structures for the language L in such a way that we can then define satisfaction functions on the resulting non-standard structures which will yield the sets of sentences of L demanded by our previous theory. Given model structures $\mathcal{M}_1$ and $\mathcal{M}_2$, and the sentences of L they generate according to the Tarskian definition—$T(\mathcal{M}_1) =_{df.} \{f \in L / \mathcal{M}_1 \models f\}$ similarly for $T(\mathcal{M}_2)$ —we want to define operations $\cup\!\!\!\cdot$ and $\cap\!\!\!\cdot$ so that:

(R) $T(\mathcal{M}_1 \cup\!\!\!\cdot\, \mathcal{M}_2) = T(\mathcal{M}_1) \cup T(\mathcal{M}_2)$ and $T(\mathcal{M}_1 \cap\!\!\!\cdot\, \mathcal{M}_2) =$
$T(\mathcal{M}_1) \cap T(\mathcal{M}_2)$.

Our discussion of the difference between schematization and superposition on the one hand and set-theoretic union and intersection on the other reminds us that the operations on the left-hand sides of these two equations are the algebraic operations of superposition and schematization, while the operations on the right-hand sides of these equations refer to the set-theoretic union and intersection of sets of sentences. The semantic effect of unions and intersections of sets of sentences requires, as we saw, some operations other than set-theoretic union and intersection of the structures by means of which we interpret L. We can expect the structures which result from these operations to look somewhat different from the standard model structures out of which they are constructed. For the sets of sentences which the non-standard structures will be built to satisfy (in accordance with our equations above) are not the consistent and complete sets which standard structures are tailored to satisfy.

In particular, there are three major respects in which the non-standard semantics we have been developing differs from the standard account, and which we may expect to find reflected in algebraic differences in the non-standard structures themselves, as well as in the account of the notion of satisfaction appropriate to them. First, of course, is the notion of identity. Our whole reason for seeking to develop a detailed model theory for inconsistent and incomplete sets of sentences is our desire to see how identity, individuation, and quantification over the elements of superposed and schematized worlds might be worked out. Next, the non-standard structures are going to have to handle conjunction in some way very different from the Tarskian fashion exhibited above in clause (iii). For in superposed worlds we may have $\mathcal{M} \models_{\bar{x}} f_1$ and $\mathcal{M} \models_{\bar{x}} f_2$ and still have $\mathcal{M} \not\models_{\bar{x}} f_1 \,\&\, f_2$ (if f_1 was true in one of the superposed standard worlds and f_2 was true in the other, but the conjunction was true in neither). Finally, negation must be handled differently than in the Tarskian scheme, since we want our non-standard structures to be able to satisfy f as well as $\sim f$, (though never f & $\sim f$), or to satisfy neither of them. Each of these respects of non-standardness of superposed and schematized structures will result in differences of structure and of the definition of satisfaction for our non-standard structures.

Let $M_1 = \langle A_1, R_1^1, R_2^1 \ldots R_n \ldots \rangle$ and $M_2 = \langle A_2, R_1^2, R_2^2, \ldots R_n \ldots \rangle$ be two standard model structures. We define a new structure M^{12}, the *superposition* of M_1 and M_2 as follows:

(1) The *domain* A^{12} is defined as the set of superposed *element-specifications* $\{x \dot\cup y / x \in A_1 \text{ and } y \in A_2\}$. A^{12} is thus a special kind of cross-product of the domains of the superposed structures. We have not yet defined the symbol '$\dot\cup$' as it appears between terms referring to individuals (rather than worlds). The expression will derive its significance from the way in which element-specifications involving it behave with respect to relations. We can treat these element-specifications just as ordered pairs of elements of A_1 and A_2, using the superposition symbol merely to remind ourselves that the ordered pairs which are superposed element-specifications will be distinct from those which are schematized ones.

(2) An explicit *identity* relation I^{12} is defined on pairs of element-specifications from the domain A^{12} by $\langle x_1 \dot\cup y_1, x_2 \dot\cup y_2 \rangle \in I^{12}$ iff $x_1 = x_2$ *or* $y_1 = y_2$. If $x_1 = x_2$, we say that $\langle x_1 \dot\cup y_1, x_2 \dot\cup y_2 \rangle \in I^{12}/1$, and if $y_1 = y_2$, that $\langle x_1 \dot\cup y_1, x_2 \dot\cup y_2 \rangle \in I^{12}/2$ Similarly, an explicit *non-identity* relation $\not{I}^{12}$ is defined on pairs of element-specifications (taking $\not{I}^{12}/1$ and $\not{I}^{12}/2$ as the complements of $I^{12}/1$ and $I^{12}/2$ respectively) by $\langle a,b \rangle \in \not{I}^{12}$ $(a,b \in A^{12})$ iff $\langle a,b \rangle \in \not{I}^{12}/1$ *or* $\langle a,b \rangle \in \not{I}^{12}/2$. It is in order to investigate the individuation of objects in non-standard worlds that we are interested in an explicit *algebraic* notion of superposition and schematization satisfying the semantic constraints of our previous discussion, so these definitions of identity and non-identity of element-specifications are the crux of the model theory we will present here.

(3) Finally, for each pair of corresponding relations R_n^1, R_n^2 of M_1 and M_2, we have their superposition R_n^{12}, a relation on A^{12} defined by $\bar{z} = \langle x_1 \dot\cup y_1, x_2 \dot\cup y_2, \ldots x_k \dot\cup y_k \rangle \in R_n^{12}$ iff $\bar{z} \in R_n^{12}/1$ *or* $\bar{z} \in R_n^{12}/2$, where $\bar{z} \in R_n^{12}/1$ iff $\langle x_1 \ldots x_k \rangle \in R_n^1$ and similarly for $R_n^{12}/2$. Again, parallel to the treatment of identity above,

for each R_n^{12} or M^{12} there is a corresponding *anti-relation* $\mathbb{R}_n^{12}$, defined by $\bar{z} \in \mathbb{R}_n^{12}$ iff $\bar{z} \notin \mathbb{R}_n^{12}/1$ *or* $\bar{z} \notin \mathbb{R}_n^{12}/2$.

A few comments are in order about the structure M^{12} which consists of $A^{12}, I^{12}, \mathbb{I}^{12}$ and the various relations and anti-relations just defined. Aside from the differences in domain which it is our project to explicate, such a superposed model structure differs from standard ones in three important respects. First, every relation (including identity) has an explicit anti-relation which is not simply its complement in the appropriate set of tuples of domain elements. This corresponds to the peculiar properties of negation in non-standard worlds. Next, there is an *explicit* identity relation. Identity in the model is not to be expressed implicitly by lexicographic distinctness of the element-specifications which we write down when we specify the domain A^{12}. This feature will enable us to resolve the issues concerning the individuation of the inhabitants of non-standard worlds raised a few pages back. Finally, the superposed structures are distinguished from standard structures by their invocation of the pair of sub-relations $R_n^{12}/1$ and $R_n^{12}/2$ (whose union constitutes R_n^{12}). This feature of the non-standard model structure corresponds to the peculiar way in which conjunction operates in superposed and schematized worlds. The distinguishability of these sub-relations clearly corresponds to the possibility of "factoring" a non-standard structure into isomorphs of the standard components out of which it is constructed. This possibility is required by the semantic role these structures are to play, namely satisfying (in the case of superposition) all and only the sentences satisfied by *either* of the superposed structures. Notice, however, that $\mathcal{M}^{12}$ is defined without mentioning the *sentences* satisfied by either of M_1 or M_2. Superposition is here given an entirely algebraic characterization, which we must show to have the semantic properties of superposition as we originally introduced that notion.

We can define the algebraic *schematization* $\mathcal{M}_{12}$ of the two standard structures M_1 and M_2 in a parallel fashion. The domain A_{12} is $\{x \frown y / x \in A_1 \text{ and } y \in A_2\}$. For $a,b \in A_{12}$, $\langle a,b \rangle \in I_{12}$ iff $\langle a,b \rangle \in I_{12}/1$ *and* $\langle a,b \rangle \in I_{12}/2$, where I_{12}/j is just like I^{12}/j except for being defined on schematized element-specifications rather than superposed ones. $\langle a,b \rangle \in \mathbb{I}_{12}$ iff $\langle a,b \rangle \notin \mathbb{I}_{12}/1$ *and* $\langle a,b \rangle \notin \mathbb{I}_{12}/2$. In the same way $\bar{z} \in (\mathrm{A}_{12})^k$ is an element of R_{12}^n just in case $\bar{z} \in R_{12}^n/1$ *and* $\bar{z} \in R_{12}^n/2$, and $\bar{z} \in \mathbb{R}_{12}^n$ iff $\bar{z} \notin R_{12}^n/1$ *and* $\bar{z} \notin R_{12}^n/2$, where R_{12}^n/j is just like R_n^{12}/j except for being defined on element-specifications of A_{12} rather than those of A^{12}.

Appendix IV specifies a notion of the *satisfaction* of a sentence of a suitable formal language by non-standard structures of these two sorts, and shows that according to that notion the superposed structure M^{12} satisfies just the sentences satisfied by *either* of M_1 *or* M_2, and that the schematized structure M_{12} satisfies just the sentences satisfied by *both* of $\mathscr{M}_1$ *and* $\mathscr{M}_2$, that is, that the operations we have just defined on algebraic structures do indeed deserve to be called "superposition" and "schematization." Since, as indicated above, structural isomorphs of $\mathscr{M}_1$ and $\mathscr{M}_2$ can be recovered from $\mathscr{M}^{12}$ and $\mathscr{M}_{12}$ using the sub-relations, it may seem that no advance has been made over our original characterization of superposition and schematization by the construction and elaboration of these models. That this is not in fact the case, however, is argued by two facts. First, the algebraic construction and the proof that it has the desired semantic properties is non-trivial (look at the proofs of the two fundamental theorems of Appendix IV). Second, and more important, producing an explicit model theory is necessary for the investigation of the identity and individuation of the inhabitants of non-standard worlds. Without algebraic details, it is impossible to see how quantification over such individuals ought to proceed, and hence it is hard to say what these individuals are like. In the next section we apply the model theory sketched here to these crucial issues.

SECTION 16

Identity and Individuation

Let us examine more closely the way in which the identity and individuation of objects is treated by the worlds for which we have proven the superposition and schematization theorems (T1) and (T2). Those theorems have given us the capacity to pass from the fusion of the *worlds* we have discussed previously, and to consider the fusion of *individuals* composing those worlds. Our demonstration of (T1) and (T2) in Appendix IV showed that it is possible to talk coherently and consistently about superposed and schematized individuals. It emerged from this discussion how such individuals must be treated algebraically so as to yield models which satisfy our definitions of semantic superposition and schematization not now merely for a propositional language, but for one with individual quantifiers. And we have also shown how it is possible to construct a model-structure which satisfies all and only the sentences satisfied by either (for superposition) or both (for schematization) of two standard model-structures. Since this result applies to all the sentences of a standard first-order predicate language with identity, it manifests the coherence and consistency of talk about fused individuals as the ontological constituents of fused worlds. But this is an abstract demonstration, and in this section we will look at its consequences for ordinary talk about the identity and individuation of such individuals.

Consider first a superposed world, constructed out of a world with exactly two elements, call them u and v, and a world with exactly three elements, x, y, and z. According to our construction in the previous section, the superposition of these two worlds yields the elements $u \uplus x, u \uplus y, u \uplus z, v \uplus x, v \uplus y, v \uplus z$. It is a measure of the strangeness of such fused individuals that we may now ask how many things there are in this world, and receive an ambiguous response. At first glance, it seems that the answer must be that there are exactly six distinct individuals in the fused world, namely those listed just above. This would be true only if the elements indicated above were comprised by a standard, Tarskian model-structure in which the satisfaction of sentences involving the identity relation turns on the distinctness or indistinctness of the *specifications* of the domain elements. In the non-Tarskian models of fused worlds and individuals which we have constructed, on the other hand, domain elements which are specified differently in the model may nonetheless be identical. For in these model-structures identity is specified by a particular relation I (and its counter-relation $\bar{I}$) which may hold between notationally distinct domain elements. To give a trivial example, it is provable in the structures specified in the last section that $I\langle u \dot{\cup} x, x \dot{\cup} u\rangle$, and that $\bar{I}\langle u \dot{\cup} x, x \dot{\cup} u\rangle$ will never hold. The difference between the specification "$u \dot{\cup} x$" of a fused object, and the description "$x \dot{\cup} u$" is not a difference that makes a difference in non-standard models of our sort. Which such differences *do* affect issues of individuation, identity, and counting in non-standard worlds is of course a function of the satisfaction relation we defined on such worlds, that is, a function of how sentences are interpreted on those structures.

The point is thus that in order to answer the question of how many things there are in the fused world we have described, it is not enough simply to count distinct specifications of domain elements, as we would in standard cases. We must instead consult the relation I and $\bar{I}$. In our construction, $I(a \dot{\cup} b, c \dot{\cup} d)$ just in case in the base-worlds which are superposed to constitute this one, $a = c$, or $b = d$. In the example above, then, we have $I(u \dot{\cup} x, u \dot{\cup} y), I(u \dot{\cup} y, u \dot{\cup} z), I(u \dot{\cup} x, u \dot{\cup} z), I(u \dot{\cup} z, u \dot{\cup} x), I(u \dot{\cup} z, u \dot{\cup} y), I(u \dot{\cup} y, u \dot{\cup} x)$, because $u = u$ in its home-world. We have $I(v \dot{\cup} x, v \dot{\cup} y), I(v \dot{\cup} y, v \dot{\cup} z), I(v \dot{\cup} x, v \dot{\cup} z), I(v \dot{\cup} z, v \dot{\cup} x), I(v \dot{\cup} z, v \dot{\cup} y), I(v \dot{\cup} y, v \dot{\cup} x)$, because $v = v$ in its home-world. We have $I(u \dot{\cup} x, v \dot{\cup} x), I(v \dot{\cup} x, u \dot{\cup} x)$ because $x = x$. We have $I(u \dot{\cup} y, v \dot{\cup} y), I(v \dot{\cup} y, u \dot{\cup} y)$ because $y = y$. We have $I(u \dot{\cup} z, v \dot{\cup} z), I(v \dot{\cup} z, u \dot{\cup} z)$ because $z = z$. We also have $I(u \dot{\cup} x, u \dot{\cup} x), I(u \dot{\cup} y, u \dot{\cup} y), I(u \dot{\cup} z, u \dot{\cup} z), I(v \dot{\cup} x, v \dot{\cup} x), I(v \dot{\cup} y, v \dot{\cup} y), I(v \dot{\cup} z, v \dot{\cup} z)$. It is clear that I is symmetric and reflexive. We'll have more to say about transitivity below. But theorem (T1) proved that if a sentence is true in w_1, it is true in $w_1 \dot{\cup} w_2$ for arbitrary w_2. We may infer from this directly that every sentence *valid* in standard, consistent and complete, worlds is also valid in all superpositions of those base worlds. This is just the logical conservativism of world-fusion which we have stressed before. Since identity is symmetric, reflexive, and transitive in all standard worlds, it must be so also in superposed worlds. The difficulty which motivated our descent into the details of a model-theory for superposition and schematization, supplementing our earlier propositional account, is that in our example the statement "there are exactly two distinct, self-identical objects" must be satisfied by the superposed world, *and so must* the statement that there are exactly *three* distinct self-identical objects. (T1) tells us that this *is* the case, and we want to see how it *can* be. The two statements above, translated into the first-order language L of our construction are:

(i) $\exists v_1 \exists v_2 \forall v_3 (v_1 \neq v_2 \,\&\, (v_3 = v_1 \vee v_3 = v_2))$

(ii) $\exists v_1 \exists v_2 \exists v_3 \forall v_4 (v_1 \neq v_2 \,\&\, v_2 \neq v_3 \,\&\, v_3 \neq v_1 \,\&$
$(v_4 = v_1 \vee v_4 = v_2 \vee v_4 = v_3))$.

Let's examine (i) first. Take as the value of v_1 the quasi-object $u \cup x$ and for the value of v_2 the quasi-object $v \cup x$. According to this assignment of values of variables, $v_1 \neq v_2$ (as in (i)), because $I(u \cup x, v \cup x)$. For recall that by our construction $I(a \cup b, c \cup d)$ just in case in the base worlds $a \neq c$ or $b \neq d$. In our example, since $u \neq v, I(u \cup x, v \cup x)$. Of course, we saw above that since $x = x$ in the base world, $I(u \cup x, v \cup x)$ as well. In superposed worlds, then, there are some pairs of quasi-objects such that *both* I and I (semantically interpreted as identity and non-identity) hold of them. This is strange, to be sure, but it is a situation we have become used to in superposed worlds for other relations than identity. It is a consequence of our separation of the evaluation and interpretation of not-*p* from that of *p*, which we have investigated at some length in the foregoing. On the above assignment of values to variables, then, $v_1 \neq v_2$. Furthermore, no matter what value we assign to v_3, either $v_3 = v_1$ or $v_3 = v_2$. For every element $a \cup b$ of the superposed world has $a = u$ or $a = v$, because of the way we constructed that world. If $a = u$, then $I(a \cup b, u \cup x)$, that is $v_3 = v_1$. If $a = v$, then $I(a \cup b, v \cup s)$, that is, $v_3 = v_2$. Thus (i) is satisfied in our example. For (ii), interpret v_1 as $u \cup x$, v_2 as $u \cup y$, and v_3 as $u \cup z$. Then $I(u \cup x, u \cup y)$, that is $v_1 \neq v_2$, since $x \neq y$. $I(u \cup y, u \cup z)$, that is $v_2 \neq v_3$, since $y \neq z$. $I(u \cup x, u \cup z)$, that is $v_1 \neq v_3$, since $x \neq z$. But now for any value $a \cup b$ we assign to v_4, either $b = x$ or $b = y$ or $b = z$, because of the way the superposition world (and its inhabitants) was constructed out of the base worlds. If $b = x$, $I(u \cup x, a \cup b)$, that is $v_1 = v_4$. If $b = y$, $I(u \cup y, a \cup b)$, that is $v_2 = v_4$. If $b = z$, $I(u \cup z, a \cup b)$, that is $v_3 = v_4$. Thus condition (ii) is satisfied in the superposition world of our example, just as (T1) assured us it would be.

But we may still be unsatisfied. How many things are there *really* in the superposed world? This is a question without an answer, or, perhaps more disturbingly, a question with too many answers. There are basically two different ways of counting the things in our superposed world. We may arrange our quasi-objects into a matrix depending upon their construction by fusion of our basic elements:

$$\begin{array}{lll} u \dot\cup x & u \dot\cup y & u \dot\cup z \\ v \dot\cup x & v \dot\cup y & v \dot\cup z \end{array}$$

The elements of the u-row are identical to each other in virtue of their shared component (i.e., $I(u \dot\cup x, u \dot\cup y)$ etc.). Similarly, the elements of the v-row are identical to each other. On the other hand, the elements of the x-column are identical to each other ($I(u \dot\cup x, v \dot\cup x)$) because of *their* shared component. The elements of the y-column are similarly identical to each other, as are the elements of the z-column. On the other hand, the elements of the u-row are distinct ($I(u \dot\cup x, u \dot\cup y)$ etc.) in virtue of their distinct second components, as are the elements of the v-row. Similarly, the elements of the columns are distinct. These facts about identity and distinction in the superposed world dovetail so as to satisfy *both* of the counting formulae (i) and (ii) of the standard base worlds. In a consistent and complete structure with Tarskian satisfaction, both those counting formulae *can't* be satisfied by one structure. No such restriction applies to the non-standard structures we have constructed.

The following objection to this state of affairs seems natural: If $u \dot\cup x$ is identical with $u \dot\cup y$ in virtue of their shared first component, *and* $u \dot\cup y$ is identical to $v \dot\cup y$ in virtue of their second component, *and* identity is transitive in the superposed world just as it is in the base worlds, $u \dot\cup x$ must be identical with $v \dot\cup y$, even though these have *no* components in common. Put another way, how can we claim that there is more than one object in the superposed world of our example, given the transitivity of identity, if it is the case that for any elements α, β of the superposed world there is some element γ such that on the one hand $\alpha = \gamma$, and on the other $\gamma = \beta$? (Our decision to treat non-identity as a matter independent of identity is irrelevant to this issue.)

The elements of the superposed world do *not* collapse in this way into a single element, however, even though identity remains transitive. The transitive law we care about is formulated in the first-order language L as $\forall v_1 \forall v_2 \forall v_3 ([v_1 = v_2 \;\&\; v_2 = v_3] \rightarrow v_1 = v_3)$. (T1) assures us that this sentence is satisfied by our exemplar superposed world, and indeed that it is valid in the class of all superpositions of standard consistent and complete worlds. The objection above seems to present a counter-example to this claim. What in fact happens if we take the value of v_1 to be $u \dot\cup x$, the value of v_2 to be $u \dot\cup y$, and the value of v_3 to be $v \dot\cup y$? The objection above urges that since $I(u \dot\cup x, u \dot\cup y)$, $v_1 = v_2$; since $I(u \dot\cup y, v \dot\cup y)$, $v_2 = v_3$; and since it is not the case that $I(u \dot\cup x, v \dot\cup y)$ as we have constructed I, the transitivity formula above fails in this instance. But clause (iii) of the definition of satisfaction for non-standard structures tells us that the fact that $v_1 = v_2$ is satisfied by a certain assignment $\bar{x}$, together with the fact that $v_2 = v_1$ is satisfied by that same assignment, does *not* suffice for $(v_1 = v_2 \;\&\; v_2 = v_3)$ to be satisfied by that assignment. That clause specifies instead that only if the *conjunction* was satisfied by assignment $\bar{x}/j$ in the base world j out of which the superposed world was constructed (that is, if the j-components of the sequence of quasi-objects which are the assignments to elements of the superposed world satisfied the conjunction). This has the effect of limiting the application of the transitive law of identity to the algebraic specifications of quasi-objects. If we have $I(a \cup b, c \cup d)$ and we have $I(c \cup d, e \cup f)$, this will entail that $I(a \cup b, e \cup f)$ only under special circumstances. If $a = c = e$ in w_1 or $b = d = f$ in w_2, then $I(a \cup b, e \cup f)$ will hold also. But if, for instance, $I(a \cup b, c \cup d)$ because $a = c$ in w_1, and $I(c \cup d, e \cup f)$ because $d = f$ in w_2, then it will *not* be the case that $I(a \cup b, e \cup f)$. It is just this last situation which the objection above envisaged. Because of the peculiar way in which conjunction is handled in non-standard structures, the case forwarded in objection does not in fact violate the transitive law for identity. $u \cup x$ and $v \cup y$ are genuinely distinct elements of the superposed world. There is no point of view from which they are identical (it is not the case that $I(u \cup x, v \cup y)$).

Thus we can divide the elements of the superposed world up into classes depending upon the relations I and $\not{I}$. In one class go elements α,β such that $I(\alpha,\beta)$ and it is not the case that $\not{I}(\alpha,\beta)$. Into this class will go pairs like $(u \cup x, u \cup x)$. Then there are elements α,β such that $\not{I}(\alpha,\beta)$ and it is not the case that $I(\alpha,\beta)$. In this class will go pairs like $(u \cup x, v \cup y)$, $(u \cup x, v \cup z)$, $(v \cup x, u \cup y)$ etc. Finally, there are elements α,β such that *both* $I(\alpha,\beta)$ *and* $\not{I}(\alpha,\beta)$. These are pairs like $(u \cup x, v \cup x)$ which are substitution instances both of $v_1 = v_2$ and of $v_1 \neq v_2$. The special definition of conjunction ensures that pairs in this last category are *not* also substitution instances of the contradiction $v_1 = v_2 \;\&\; v_1 \neq v_2$. It is the existence of this third class of pairs of elements which makes identity an occasionally indeterminate matter in superposed worlds. The existence of the other two classes, together with the special reading of conjunction which assures that all logical laws (read collectively or conjunctively) are valid in superposed worlds, keep identity from being a *completely* indeterminate affair.

The objection above is thus based on a failure to read the transitivity of identity in the conjunctive way which our approach requires and which our model-theory incorporates. There is a way of counting the things in the superposed world so that there are exactly two of them. $u \uplus x = u \uplus y = u \uplus z$ is one object, (the transitive law, in the form which *does* hold for these objects, entitles us to this way of writing things) and $v \uplus x = v \uplus y = v \uplus z$ is the other. There is another way of counting things in the superposed world so that there are exactly three of them. $u \uplus x = v \uplus x$ is one of them, $u \uplus y = v \uplus y$ is another, and $u \uplus z = v \uplus z$ is the third. There is no way of counting the things in the superposed world in such a way that there are fewer than two or more than three objects. The objects in the superposed world are partly indeterminate as to their identity and individuation, but not completely so. Thus the general characteristics of our account of superposed *worlds*—mutually contradictory facts are true of them, but no self-contradictory ones, in virtue of their construction by fusions out of standard consistent worlds—extend straightforwardly to *individuals* created by the superposition of standard consistent individuals.

Consider now the individuals which inhabit *schematized* worlds, and so how identity and individuation are treated by our model-theory. Take as an example a world created through the fusion by schematization of the following two worlds: W_1 has just the two objects u,v in it, while W_2 has the four objects w,x,y,z as its domain. The schematized world which results has as its elements:

$$\begin{matrix} u \Cap w & u \Cap x & u \Cap y & u \Cap z \\ v \Cap w & v \Cap x & v \Cap y & v \Cap z \end{matrix}$$

How many things are there in this world? To begin with, in W_1 the statement "There are at least two things," is true, and this same statement is true in W_2. Similarly, in W_2 it is true that there are not more than four things, and this statement also holds for W_1. By the schematization theorem (T2), then, both these statements will be true in $W_1 \Cap W_2$. That is,

(i) $\exists v_1 \exists v_2 (v_1 \neq v_2)$ and

(ii) $\forall v_1 \forall v_2 \forall v_3 \forall v_4 \forall v_5 ((v_1 \neq v_2 \mathbin{\&} v_2 \neq v_3 \mathbin{\&} v_3 \neq v_1 \mathbin{\&} v_3 \neq v_4 \mathbin{\&} v_4 \neq v_1 \mathbin{\&} v_4 \neq v_2) \rightarrow (v_5 = v_4 \vee v_5 = v_3 \vee v_5 = v_2 \vee v_5 = v_1))$.

So it has to come out true in $W_1 \Cap W_2$ that there are at least two things, and not more than four things. But matters are more difficult than this. (T2) says that $W_1 \Cap W_2$ will satisfy all and only the sentences which are satisfied by *both* W_1 and W_2. This means that the sentence which says that there are exactly two things cannot be true in $W_1 \frown W_2$, for that is not true in W_2. Nor is the sentence which says that there are exactly four things true in $W_1 \frown W_2$, for it is not true in W_1. The claim that there are exactly three things, of course, is not true in either W_1 or W_2, and *a fortiori* not in $W_1 \frown W_2$. So while in $W_1 \frown W_2$ it is true that there are between two and four things, it is not true either that there are exactly two, exactly three, or exactly four. The number of objects in $W_1 \frown W_2$ is not completely indeterminate, but it is not determinate either. Looking at the negations of these sentences, we find that the statement that there are not exactly three things is true in $W_1 \frown W_2$, since it holds in both the base worlds. But the claims that there are not exactly two and not exactly four things do *not* hold in $W_1 \frown W_2$, for each fails in a base world. This is indeed a peculiar situation, and it is difficult to imagine how the fused individuals must behave in order to satisfy these constraints, as (T2) assures us they do.

In superposed worlds we were faced with pairs of element specifications (for instance $u \smile x$, $u \smile y$) such that *both* the identity and the non-identity relations obtained between them. It was these only ambiguously distinct or identical pairs which accounted for the controlled indeterminacy of individuation we observed there. In schematized worlds we are faced with a somewhat different situation. Here there can be no pairs of element specifications of which both identity and non-identity hold. Rather we find pairs of such specifications for which *neither* identity nor non-identity relations obtain. Thus in our example above, consider the pair $u \frown w$ $v \frown w$. By our construction, $I(a \frown b, c \frown d)$ just in case $a = c$ *and* $b = d$ and $\not{I}(a \frown b, c \frown d)$ just in case $a \neq c$ *and* $b \neq d$. In the pair above, $u \neq v$ and $w = w$, so neither I nor $\not{I}$ hold. Indeed, none of the eight element-specifications in the schematized world we are considering are identical to each other, since each differs from every other by at least one component. There are, however, pairs of element-specifications to which non-identity properly applies. Thus $\not{I}(u \frown w, v \frown x)$, $I(u \frown x, v \frown z)$ and so on, for any pairs which do not share at least one component. The exis'ence of this last category of pairs of element-specifications of the model ensures that numbering statement (i) above, claiming that there are at least two distinct things, holds in the schematized world. Let the value of v_1 be $u \frown z$, and the value of v_2 be $v \frown w$, and since $\not{I}(u \frown z, v \frown w)$ sentence (i) is satisfied.

To see how the second numbering statement, the claim that there are not more than four things, gets satisfied in the schematized world. Consider the pairs for which I holds. These pairs of element-specifications must consist of four different components, as with $u \frown x, v \frown y$. But there are only two component-specifications available for the first place of each element-specification. Each element must be a fusion of u with something, or a fusion of v with something, since these are the only objects in W_1. This means that whenever there are α,β,γ, such that $I(\alpha,\beta)$ and $I(\alpha,\gamma)$, it is *not* the case that $I(\beta,\gamma)$. For α and β must have different initial component-specifications (if α is a u-fusion product, β must be a v-fusion product, and vice versa), and the same must be true of α and γ. Thus β and γ must have the same initial component-specification (v if α is a u-fusion product and u if α is a v-fusion product). By the construction of I, then, it cannot be the case that $I(\beta,\gamma)$. This means that the antecedent of the conditional in (ii) will never be satisfied in $W_1 \frown W_2$, for that antecedent specifies that $v_1 \neq v_2$ & $v_2 \neq v_3$ & $v_3 \neq v_1 \ldots$ and we have just seen that there is no assignment of element-specifications to v_1,v_2,v_3 which can meet this condition. Thus the conditional assertion (ii) is vacuously satisfied in $W_1 \frown W_2$.

In $W_1 \frown W_2$, then, we can be sure that there are at least two different things, for we can pick a pair of element-specifications which pick out distinct quasi-objects, for instance $u \frown w, v \frown x$. To continue counting in a determinate fashion, however, we must be able to tell of any third element-specification whether it is identical to, or distinct from each of these initial ones. We can pick another element-specification which will be distinct from *either* of the initial two distinct ones (since, e.g., $I(u \frown w, v \frown y)$, and $I(v \frown x, u \frown y)$) but we can never find one which is distinct from *both* (thus it is not the case that $I(u \frown w, u \frown y)$, nor is it the case that $I(v \frown x, v \frown y)$). Pick the initial distinct pair how you like, the third candidate examined will be indeterminate as to its identity and difference from one of the members of that initial pair. The identity and distinctness relations which we must apply to element-specifications in order to determine what and how many quasi-objects there are in a fused world created by schematization of two standard worlds are thus vague in precisely the fashion required to yield the conclusion that there are at least two, no more than four, and not exactly three quasi-objects in the non-standard world (our argument showing that the antecedent of the conditional in the numbering formula (ii) above can't be satisfied in $w_1 \frown w_2$ showed that there were not exactly three distinct things in $w_1 \frown w_2$).

SECTION 17

Object Stipulation

The things which make up non-standard worlds are accordingly not to be identified with the element-specifications which we write down when we are describing such worlds, but the quasi-objects which those specifications, together with the relations I and $\not{I}$, determine. Because of the partially indeterminate character of the identity and individuation of these quasi-objects, we cannot simply write down names for them when we specify a model for the set of sentences satisfied in a non-standard world. For how many names would we use? Our endeavor all along has been to use a standard, consistent and complete metalanguage to discuss non-standard, inconsistent and incomplete worlds and the (quasi-)objects they comprise. Thus, even were it possible, we have no motivation whatsoever to describe non-standard worlds using a language containing names whose identity and individuation as syntactic objects is ambiguous or indeterminate in the same way as are the quasi-objects they refer to. Instead, in speaking about these strange entities we have had to refer to them in a more circuitous fashion which avoids direct naming. When specifying the contents of a standard possible world, one can simply name the objects in it (at least if there aren't too many), and define the relations which hold among them by using those names. To specify a non-standard world and its inhabitants in a consistent and complete standard language, however, the device of element-specifications and explicit identity and non-identity relations must be employed. The use of such a device does not entail that we cannot refer directly to the quasi-objects which make up non-standard worlds. The variables and open formulae of the object language L designate the quasi-objects, for instance. Our examples of the way in which identity behaves in non-standard worlds so as to satisfy the superposition and schematization theorems (T1) and (T2) showed that it is quasi-objects with ambiguous identity and non-identity relations that the language L, interpreted according to our non-standard satisfaction relation, is talking *about*. And even at the level of our working metalanguage, within which element-specifications are formulated, direct reference to quasi-objects is achieved. The expression '$u \odot w$,' for instance, refers to a quasi-object. The only trouble is that while it is true that it refers to a different quasi-object than '$v \odot x$' does, the quasi-object referred to by '$u \frown x$' is neither identical to either of these, nor is it distinct from them. Such a situation means that the Fregean desideratum "One object, one name" must be foregone. But it does *not* make reference, denotation, or designation impossible.

In this connection it is worth considering an objection which might be raised against our account of the quasi-objects which inhabit non-standard possible worlds. The objection begins by noting an analogy between the way in which we have proceeded in our treatment of intra-world identity and individuation, and recent discussions of trans-world identity and individuation. Kripke, for instance[82] describes a common way of thinking about individual identity across possible worlds:

> One thinks, in this picture, of a possible world as if it were a foreign country. One looks upon it as an observer. Maybe Nixon has moved to the other country and maybe he hasn't, but one is given only qualities. One can observe all his qualities, but of course, one doesn't observe that someone is Nixon.

A criterion of identity is supposed to resolve this difficulty by giving necessary and sufficient qualitative conditions for an object presented in some possible world being the individual Nixon. But, Kripke correctly argues, this view of possible worlds is mistaken:

> A possible world isn't a distant country that we are coming across or viewing through a telescope . . . A possible world is *given by the descriptive conditions we associate with it* . . . Why can't it be part of the *description* of a possible world that it contains Nixon . . . ? . . . 'Possible worlds' are *stipulated*, not *discovered* by powerful telescopes.

Although he does not argue further in this way, he might have continued by pointing out that we *stipulate* the trans-world identity of the *qualities* which set the problem of individual identity, even according to the telescope view. One must take for granted the capacity to tell that the property of being red or being human, or being President, is exemplified in a distant world (or some similar set of properties), lest other possible worlds become entirely indeterminate. But then why should we stipulate the properties which may be found in other worlds, and stick at stipulating the objects?

We raise this point here because it might be thought that the way in which we have handled *intra*-world identity in non-standard possible worlds commits precisely this telescope fallacy. For we presented in our metalanguage descriptions of the furniture of superposed and schematized worlds *first* a set of element-specifications of the form $a \smile b$ or $a \frown b$, and *second* a pair of relations, I and I, applying to the element-specifications and in terms of which criteria (albeit partially ambiguous) of identity and individuation of quasi-objects were formulated. Why, it may be asked, should we not simply *stipulate* what things and how many there are in a given non-standard world, instead of being *given* element-specifications and the relations *I* and *I* and having to use them to *discover* what things and how many there are in that world? Does not our account of the identity and individuation of quasi-objects present us in the guise of Kripke's travellers to a foreign country who must infer what individual objects there are from what can be directly apprehended, together with special criteria of identity? That this procedure is required even within the confines of a single world, instead of involving identity across worlds, surely makes it no more acceptable.[83]

Kripke's point is that specifying a possible world *is* specifying the things in it, and that the requirement that this be done in the particular fashion of offering qualitative descriptions and then defining equivalence classes of them is perfectly gratuitous. Now we have indeed specified the things in non-standard worlds by offering preliminary designations (the element-specifications) and defining equivalence classes of them (though the equivalence classes behave somewhat strangely because transitivity is read collectively rather than distributively, and because non-identity is not taken to be simply the complement of identity in the domain of element-specifications). But in our case this procedure is *not* gratuitous. It is the only way in which quasi-objects (things whose identity and individuation behave in the ways we have detailed above) *can* be specified. As we noted above, the partial indeterminacy of the identity and individuation of quasi-objects precludes simple naming of them in a standard language in which the conditions of identity and individuation of the *expressions* which do the naming are completely determinate. The preliminary designations with respect to which an explicit identity relation is defined have nothing to do with the qualitative apprehension of quasi-objects. They are simply a syntactic device in our metalanguage to enable us to talk determinately about quasi-objects, in spite of the indeterminacy which infects them. Granting the necessity of this move, our stipulation of the things inhabiting non-standard possible worlds is straightforward and unobjectionable. The reference of our metalinguistic expressions must be determined *holistically*, by looking at the element-specifications *and* the explicit identity and non-identity relations. This contrasts with *atomistic* reference achieved by constants and variables in a standard metalanguage without requiring the assistance of other parts of the model.

It is worth pointing out that this indirect mode of reference via element-specifications and explicit identity and non-identity relations does *not* require us to have an expression in the meta-language in which we do our model-theory for every domain element (which would restrict us to countable domains). For we can say in that metalanguage: $(\forall x \in A_1)(\forall y \in A_2)(x \frown y \in A_{12})$, and this expression is to be interpreted as invoking *referential*, and not merely substitutional quantification over both the elements of the standard domains A_1 and A_2 *and* the non-standard domain A_{12}.

SECTION 18

The Lattice of Possible Worlds

The manifold of non-standard possible worlds contains an element of structure which its basis, the realm of standard worlds, does not. For each element in this domain is constructed in a definite way by superposition and schematization, out of other elements. In this section we will give a precise mathematical characterization of this added element of structure, and in the sequel we will exploit it to show how the whole apparatus of non-standard entities can be turned to the philosophical purpose of an analysis of belief and cognitive inquiry. Finally, we will use some of the details of this application of the structure of inconsistent and incomplete entities to argue for the ontological *parity* of standard and non-standard individuals and worlds.

Non-standard possible worlds as we have defined them can be made to form a mathematical *lattice* under the operations of superposition and schematization. To see this, consider the set of all standard algebraic structures which are models of theories of some first-order language L, according to the standard, Tarskian account of satisfaction. Call these non-standard worlds of *constructive degree zero*. The set of non-standard worlds of constructive degree one is the set of all non-standard worlds which are the result of schematizing or superposing together any set of non-standard structures of constructive degree zero. In general, the set of non-standard worlds of constructive degree $n+1$ is the set of superpositions and schematizations of subsets of the set of non-standard worlds of constructive degree less than or equal to n. The set of non-standard structures is the set of all non-standard structures of any constructive degree.

We will construct a lattice on this domain of non-standard possible worlds (which we have just seen includes the standard worlds), using the algebraic operations of superposition and schematization which we have defined on that domain. Thus given any two elements a, b of the set of non-standard possible worlds, that set contains also $a \uplus b$ and $a \Cap b$. We cannot simply identify our operations of superposition $\uplus$ with the lattice-theoretic "join" operation, and our operation of schematization $\Cap$ with the lattice-theoretic "meet" operation, however. To form a lattice, the operations must be shown to meet the following four conditions[87] (of which the first is a consequence of the last three, which are mutually independent).

L1: $a \frown a = a, \quad a \smile a = a$
L2: $a \frown b = b \frown a, \quad a \smile b = b \smile a$
L3: $a \frown (b \frown c) = (a \frown b) \frown c, \quad a \smile (b \smile c) = (a \smile b) \smile c$
L4: $a \frown (a \smile b) = a \smile (a \frown b) = a$

Non-standard structures as we have defined them do not quite satisfy these conditions, but we can easily transform the set of worlds which doesn't meet these conditions into one which does. We do this by defining an equivalence relation on the original set which is *stipulated* to satisfy L1–L4, and then work on the lattice of equivalence classes under that relation. Such a stipulation requires some care, however, so let's look a little more closely at the problem.

Our major concern is to keep the semantic consequences of our algebraic construction (as summed up in (T1) and (T2)) intact. In Appendix IV we saw how the semantic properties of our construction could be retained through various sorts of generalizations of the construction for which we gave detailed demonstrations of the central theorems. We must be sure that the same properties will be retained in the lattice generated by shrinking the domain of non-standard possible worlds to equivalence classes according to L1–L4, and using the meet and join operations induced on those equivalence classes by schematization and superposition in the original domain. What are those semantic properties? If we represent the set of sentences satisfied (non-standardly) by a structure w as Tw, the condition is that

(1) $T(w_1 \smile w_2 \smile w_3 \smile \ldots \smile w_n \ldots) = Tw_1 \cup Tw_2 \cup \ldots Tw_n \cup \ldots$
and
(2) $T(w_1 \frown w_2 \frown w_3 \ldots \frown w_n \ldots) = Tw_1 \cap Tw_2 \cap \ldots Tw_n \cap \ldots$

Consider for the moment just the entities which we operate on the right side of these equations. These are sets of sentences, which we union and intersect at will, and all are subsets of the set of all sentences of L (we can describe them better than that, as we will see below). Such subsets clearly form a lattice under set-theoretic union and intersection. What we want to do is to construct a homomorphic lattice of non-standard possible worlds such that when we *shrink* the superposition–schematization lattice of non-standard algebraic structures according to the equivalence relation "(non-standardly) satisfies the same set of sentences as", we get just the semantic lattice defined by union and intersection of sets of sentences. Our constructions so far do not enable us to do this, even with the generalizations we sketched, since algebraic superposition and schematization don't meet the lattice axioms L1–L4. However, what we propose to do is to take an equivalence relation stipulated to satisfy those axioms, shrink the set of non-standard possible worlds according to *that* relation, and then establish the desired homomorphism between the resulting lattice of equivalence classes and the semantic lattice. This procedure will accordingly be justified just in case the equivalence relations needed to convert our set of non-standard worlds with algebraic operations of superposition and schematization into a genuine lattice are *restrictions* of the equivalence relation "(non-standardly) satisfies the same set of sentences as". In short, we can simply *stipulate* that, e.g., commutativity (L2), holds for non-standard structures, just in case $T(w_1 \cup w_2) = T(w_2 \cup w_1)$ (and dually).

This condition turns out to be satisfied easily. Although our notation as presented so far distinguishes between w and $w \cap w$ and $w \cup w$, the generalized forms of our major theorems (T1) and (T2) show us that $T(w \cap w) = Tw = T(w \cup w)$. Thus we are entitled to consider w, $w \cap w$, and $w \cup w$ as all one non-standard possible world, specified in various (and variously perspicuous) ways. Indeed, it is clear that each of the identities asserted by L1–L4 above are asserted between non-standard worlds which satisfy the same sets of sentences of L, and are thus acceptable.

More specifically, we define a relation R which is symmetric, transitive, and reflexive, and such that the following eight relations hold for all w_1, w_2, and w_3, non-standard worlds of any constructive degree:

$$R(w_1 \cap w_1, w_1),\ R(w_1 \cup w_1, w_1),\ R(w_1 \cap w_2, w_2 \cap w_1),$$
$$R(w_1 \cup w_2, w_2 \cup w_1),\ R(w_1 \cap (w_2 \cap w_3), (w_1 \cap w_2) \cap w_3),$$
$$R(w_1 \cup (w_2 \cup w_3), (w_1 \cup w_2) \cup w_3),$$
$$R(w_1 \cap (w_2 \cup w_1), w_1 \cup (w_2 \cap w_1)),\ R(w_1 \cap (w_2 \cup w_1), w_1).$$

Henceforth the expression "(non-standard) possible world" is to be understood as referring not to such worlds as they have been defined up to now, but to the equivalence classes of such worlds defined by the relation R. The stipulations specifying R above are tailored to make the domain of possible worlds (equivalence classes under R) thus defined into a lattice, according to conditions (L1)–(L4).

But can we simply identify a world with a *class* of worlds? What is the significance of such a stipulation, aside from the nice formal structure which results from it? Consider lattice axiom L2, commutativity. It is clear that our introduction and motivation of the operations of superposition and schematization did not involve assigning any significance to the *order* in which two worlds are fused. It is merely an unfortunate side-effect of the notation we use to specify the product of such a world-fusion that two different such specifications ($w_1 \frown w_2$ and $w_2 \frown w_1$) are possible. Clearly stipulating that these two world specifications refer to the *same* world (which is the effect of turning our attention to the R-shrink of our original domain) merely undoes a spurious distinction otherwise enforced by an unperspicuous notation. Similarly, lattice axiom L3, associativity, also undoes a notational infelicity. Nothing in our motivating remarks suggested that the schematization (or superposition) of three worlds had to proceed in any particular order, or even had to proceed by the fusion of *pairs* of worlds at all. Our previous notation provided two different ways of specifying what is intuitively *one* world, $w_1 \frown w_2 \frown w_3$. Our current stipulation just removes this misleading suggestion.

The remaining two lattice-axioms, the absorption axioms L1 and L4, resolve another such difficulty. Our original motivating remarks concerning the ontological operations of schematization and superposition envisaged the results of these operations as worlds where, respectively, all and only those situations obtained which obtained in *both* of the relevant base-worlds, and where all and only those situations obtained which obtained in *either* of the base-worlds. The superposition or schematization of a world with *itself* ought then just to be that world again. Further, $w_1 \smile (w_1 \frown w_2)$ ought to consist of those states of affairs which obtain in w_1 *or* obtain in w_1 *and* in w_2, namely just those which obtain in w_1.

In fact these stipulations make less difference than at first might appear to be the case when we think of them as replacing single worlds with equivalence classes of worlds. In fact the equivalence relations holds between world-*designations* just in case all refer to a single world. In the case of the equivalence of specification induced by stipulation L2 and L3, we simply agree not to distinguish worlds with mutually commutative or associative specifications. In the case of L1 and L4, we can simply take the world w_1 to be the *canonical representative* of the equivalence class of world-designations which includes $w_1 \frown w_1$, $w_1 \smile w_1$, $w_1 \frown (w_1 \smile w_2)$, $w_1 \smile (w_1 \frown w_2)$, and so on.

It is important to notice that we are thus semantically entitled to stipulate the identities L1–L4 only at the level of *worlds*. The counterpart principles, applied to the *individuals* in those worlds will not apply. Thus we may not stipulate that every element specification of the form $x_1 O_1 x_2 O_2 \ldots x_n O_n x_{n+1} O_{n+1} \ldots$ (each $O_i = \smile$ or $\frown$)

where $x_n = x_{n+1}$ can be replaced by one of the form $x_1 O_1 x_2 O_2 \ldots x_n O_{n+1} x_{n+2} \ldots$ without violating our semantic principles. In this case, some element-specifications would have fewer components than others, and our definition of satisfaction presupposes that this was not the case. Nor can we specify even that when *every* element-specification in a given domain has such a redundancy at the n^{th} and $n+1^{st}$ places, that *all* should be replaced as above. For this would reduce $w_1 \cap w_2$ to w_1 in every case in which the base worlds w_1 and w_2 had a common *domain*. Since such structures can obviously satisfy different sets of sentences, this would be an illegitimate stipulation.

We can thus stipulate the identities L1–L4, and the resulting set of non-standard structures with the operations of superposition and schematization form a lattice. This lattice is *not* isomorphic to the semantic lattice of sets of sentences with the operations of union and intersection, since *many* non-standard structures will satisfy the same set of sentences. Thus, if we take any two non-standard worlds that satisfy exactly the same set of sentences, their schematization and superposition will be further non-standard worlds in which just the same sentences are satisfied. We can be sure that there are such pairs of non-standard worlds because there are such pairs of standard worlds, and they just *are* non-standard worlds of a special kind. Nor is this all. It is clear that any consistent and complete set of sentences will not only have standard models, but will also have *non-standard* models. Thus it is clear that the fusion (whether by schematization or superposition) of any set of standard models of some maximal consistent set **T** of sentences of a language *L* will also be a model of **T**. Again, if we take for each sentence $t \in \mathbf{T}$ the schematization of the set of all standard models which satisfy *t*, and superpose all of these sets, we will get a non-standard model of **T**. And still other constructions of non-standard models of maximal consistent theories are possible. These non-standard models have domains consisting of quasi-objects with their strange properties of identity and individuation, but in these cases *L* is not sufficiently expressive to distinguish such models from standard ones. Thus it will turn out that even if in the end some theory emerges from an inquiry as consistent and complete, this will not settle the question of the standardness of the domain it discusses. For the language in which the theory is couched may not be powerful enough to distinguish the two cases. (This situation should be compared with that in arithmetic and analysis upon Robinson's introduction of non-standard models.)[85]

In standard model theory, one seeks to connect a syntactically defined notion of *consistency* with an algebraically defined notion of *having a model.* The semantics which results is *sound* in case every set of sentences that has a model is consistent, and it is *complete* in case every set of sentences that is consistent has a model. In keeping with the terminology of section 7 above, we may call a set of sentences *minimally consistent* iff it has no logical falsehoods as members. Similarly, we may call a set of sentences *minimally complete* iff it contains all the logical tautologies of the appropriate language. Then the model theory we have developed is sound and complete with respect to the set of all sets of sentences which are both minimally consistent and minimally complete. It is sound, because every set of sentences that is satisfied by a non-standard model is minimally consistent. It is complete, because for every minimally consistent set of sentences which is also minimally complete, there is a non-standard model which satisfies *exactly* the elements of that set. This may be seen as follows. By the way in which we defined non-standard structures, each structure of constructive degree n is the result of applying certain well-specified operations to a particular set of structures of constructive degree $n-1$ (recall that each degree includes the union of all structures of smaller degrees). Since structures of constructive degree zero are standard structures, each structure of whatever degree has a definite set of standard structures out of which it is composed. Our stipulation of L1–L4 made it possible for different paths from a base set of standard structures to lead to the same non-standard structure of constructive degree *n*, but not for one such structure to be constructible starting from two different sets of standard base worlds. Algebraically, then, the standard worlds are a *basis* for the non-standard ones, with respect to the operations of superposition and schematization. Further, the generalized versions of the superposition and schematization theorems tell us how to determine what set of sentences a complex non-standard world satisfies, based on its algebraic construction out of standard worlds. So (T1) and (T2) serve to assure that our non-standard model theory will be sound and complete with respect to weakly consistent sets of sentences just in case all such sets are constructible by union and intersection of the sets of sentences satisfied by standard structures. But this last is clear. For we can construct the non-standard world in which, besides the tautologies, only the sentence s holds, by taking the intersection of all the maximal consistent sets of sentences containing s. Any desired weakly consistent set can then be constructed by taking the union of all such intersections, indexed over the sentences in the weakly consistent set. *Semantically*, the set of maximal consistent sets of sentences, and the set of one-element sets of sentences, both form bases for the minimally consistent and complete sets of sentences, with respect to the operations of union and intersection. Thus the mapping which assigns to every non-standard structure the set of sentences of L which it satisfies is a homomorphism from the lattice of non-standard possible worlds with algebraic superposition and schematization as operations into the lattice of minimally consistent and complete sets of sentences with union and intersection as operations. Calling this mapping *h*,

$$h(w_1 \uplus w_2) = hw_1 \cup hw_2 \text{ and } h(w_1 \Cap w_2) = hw_1 \cap hw_2$$

where the joins are superposition of structures on the left side of the equation, and unions of minimally consistent and complete theories on the right, and the meets are schematizations of structures on the left side of the equation, and intersections of minimally consistent and complete theories on the right.

This result, which was really implicit already in our discussion after the detailed demonstration of the initial forms of the superposition and schematization theorems, is not analogous to any result of standard model theory. For the lattice structure which is preserved by the mapping *h* (which is both a join homomorphism and a meet-homomorphism) has no standard analogue. What operations on maximal consistent sets of sentences are we to liken to superposition and schematization in the non-standard case? Union and intersection of maximal consistent sets of sentences yield such sets only in degenerate cases. Nor are there parallel, suitably closed, operations on standard algebraic structures. In standard model theory one shows an isomorphism between the set of material-equivalence classes of sentences of *L* with the partial ordering induced by the entailment relation, and the set of semantic-equivalence classes of standard models with the partial ordering of set-theoretic inclusion (The Deduction Theorem).[86] A version of this theorem can easily be proven with the materials now at hand for non-standard models as well, using collective rather than distributive readings of entailments. Our concern in the rest of this essay, however, will be with those aspects of the non-standard structures, codified in the lattice-homomorphism specified above, which *have* no standard analogue.

SECTION 19

Belief and Non-Standard Possible Worlds

The preceding discussion has shown how a consistent logic may be constructed for inconsistent and incomplete theories and how a sound and complete model theory for such weakly inconsistent and incomplete quantified theories can be produced. That model theory enables one to describe in detail the quasi-objects which inconsistent and incomplete theories are theories *of*. In the philosophical tradition, such Meinongian objects are generally thought of as *ideal* entities, objects of some propositional attitude, paradigmatically of *belief*. The next few sections will investigate the suitability of non-standard worlds for the representation of such attitudes.

Our concern throughout this essay has been with the ontological respectability of non-standard possible worlds, with their claim to citizenship in the republic of possible worlds with the same civil status as standard, consistent and complete, worlds. The consideration of the aptness of non-standard worlds for the representation of belief is not a retreat from this position. On the contrary, it will be shown that although non-standard worlds can usefully model what is *believed* to be the case, there is no reason to conclude that non-standard worlds are therefore ideal in some sense in which standard possible worlds are not. In particular, it will be argued that there is no more reason to believe that non-standard possible worlds represent *epistemic* states of affairs (e.g., merely subjective beliefs) while standard worlds alone can represent *ontic* states of affairs (e.g., how things objectively are or might be) than the other way around. Thus we will argue for *parity* of ontological status between standard and non-standard possible worlds and their inhabitants. We will show in some detail how standard worlds can be considered as the outcome of an *ideal* process of inquiry, while everything that *actually* occurs in the inquiry is represented by non-standard possible worlds. This demonstration will thus parallel our previous exhibition of non-standard worlds as the results of operations performed on standard ones. The conclusion towards which the discussion moves is that whatever ontological status is assigned to standard, consistent and complete possible worlds, ought also to be assigned to non-standard possible worlds.

Non-standard worlds are attractive as an apparatus for characterizing belief primarily because they provide a precise, formalizable semantics of minimally consistent and complete theories, and the set of beliefs held at a time by an individual is expressible by such a theory. The set of sentences believed by a speaker at a time, approximated by the set of sentences that speaker is disposed sincerely to assent to under some hypothetical standard conversational conditions, will in general meet our specifications for minimally consistent and incomplete theories, to which our non-standard meta-theory applies, and will not in general meet the specifications of a *strong* consistency within which context alone standard logic finds application. The minimal consistency and completeness condition demands that (i) all logical truths be believed, and that (ii) no logically self-contradictory thesis be believed, and (iii) that all sentences deducible from believed sentences by one-premise inferences of first-order logic is believed. Anyone whose beliefs putatively did not satisfy these three conditions, someone who is willing to assent sincerely to logical contradictions and is unwilling to so assent to tautologies or who does not believe the *immediate* consequences of his beliefs, simply offers the best evidence imaginable that his language is not the one we though it was. Far from forcing the admission of sets of beliefs which are not minimally consistent and complete, such a situation would show that the beliefs of the individual in question had not been understood.

On the other hand, intelligibility of beliefs imposes no stronger requirement than minimal consistency and completeness, since we have shown how a standardly consistent and complete meta-theory for minimal consistency can be developed. One therefore need not attribute consistent and complete sets of beliefs to an individual in order to express his attitudes coherently. It is here that the signal virtue of the analysis in terms of non-standard possible worlds for the expression of belief appears. For the paradigmatic expression of a belief is a disposition sincerely to assent to a statement when queried under standard conversational conditions, and such dispositions are typically neither consistent nor complete with respect to the statements of any language. As for completeness, even in the context of pure number theory there are conjectures whose truth or falsity has not been demonstrated and about which one may reasonably remain, not merely agnostic, but adoxastic—entirely without opinion. How much less complete are our beliefs outside of mathematics! Most of us have no views about the relative abundances of silicon and aluminium in distant galaxies, the presence or absence of red nettles in Tibet, whether the number of single-family dwellings in the

Pittsburgh city-limits is odd or even, and so on. To believe rationally by our canons is often to refuse credence both to the claim, e.g., that it will rain tomorrow *and* to the claim that it will not (though of course, in accord with the requirements of minimal consistency, one will continue to believe that it either will rain tomorrow or not, and one will not believe that it will *both* rain tomorrow and not), when there is no evidence at all available, or if the only evidence available is the conflicting testimony of two equally unreliable informants.

Nor are our dispositions to assent to statements limited to consistent sets of statements. Even when we are maximally careful about the consistency of our theories, we may be done in by their complexity. Thus even in mathematics and logic, there are such familiar examples as Frege's inconsistent second-order logic, and Quine's difficulties with his version of set theory. At a more mundane level, it is common for an individual to be simultaneously disposed to assent to a statement if queried and to be disposed to assent to the denial of that statement if queried in some variant context (once again, this does not mean that one is ever disposed to assent to the conjunction of a statement and its denial).

With our apparatus, we can express such contextual relativity of belief, representing the various contexts by different subsets of beliefs of the speaker about the circumstances of the discourse, and the different dispositions to assent as the results of conjunctive multi-premise *inferences* from those beliefs.

As long as the dispositions to assent to the statement and its denial are not made the objects of conscious scrutiny (for instance by being actualized), they can coexist in peace. Such situations are common, and are most aptly described as the holding of (minimally) inconsistent beliefs. Any inquiry in which one comes to hold new beliefs, expressed by novel dispositions to assent in conversation, leaves the inquirer with the substantive task of canvassing old beliefs and their associated conversational dispositions for statements which entail the denial of some conclusion reached in the course of the inquiry. This separate process, phenomenologically familiar as "coming to realize the significance" of some view, or "allowing a conclusion to sink in", is the attempt to rectify the grosser inconsistencies in one's beliefs which result from reaching novel conclusions. Such a process is rarely completed, and its completion is certainly not a precondition of believing the novel conclusion which was the result of one's inquiry. So although there is indeed a force which acts on our beliefs at various times, seeking to drive them in the direction of greater consistency (about which more later), it is not the case that simply adopting a statement not previously believed automatically purges one's beliefs of all those statements which in the context of some conjunction of premisses entail the denial of the novel claim. (Indeed, as Socrates taught us, one's entire intellectual energies could usefully be expended in the effort to render more consistent even our commonest beliefs.)

The phenomenon of inconsistent belief would not be worth belaboring in this fashion were it not the case that it is the source of considerable difficulty in the representation of beliefs by standard possible worlds (which model only strongly consistent sets of sentences) and of corresponding ingenuity on the part of theorists seeking to employ the standard apparatus for that purpose. The difficulty arises not so much with expressing inconsistent beliefs in the first place, as in dealing with inference. For one can represent beliefs with a *set* of standard possible worlds, in some of which a statement p holds, and in others of which its denial $\bar{p}$ holds. But one of the simplest logical inferences one can make is from a statement and its denial to any arbitrary thesis whatsoever. The standard theorist is thus faced with a dilemma: Either deny minimal logical competence to believers, or face the collapse of all inconsistent belief-sets into the identical absurd set of all the sentences of the language (self-contradictory or not). The second alternative is clearly unacceptable, as it trivializes the notion of belief beyond its fitness for any explanatory task. The first alternative is only slightly more palatable. For what is the point of representing beliefs if one cannot also represent the inferences by which they are transformed in the course of inquiry? On the other hand, beliefs which were inferentially isolated from their fellows by the complete logical insulation entailed by a denial of even minimal logical competence can play little explanatory role in an account of behaviour or inquiry. Some sort of restriction on inference must be formulated which prevents the absurdity of identifying the sort of practically uninfectious inconsistency referred to above with believing all the sentences of one's language, while retaining enough inferential capacity for beliefs to be cognitively efficacious.

Our account of non-standard deductive theory and model theory, of course, is designed precisely to resolve this dilemma. Each non-standard world satisfies the logical consequences of every thesis which it satisfies. But it will satisfy the standard consequences of a collection of statements (e.g., p, $\sim p$) *only* if their conjunction is also satisfied (as will *never* be the case in a non-standard world for p, $\sim p$). Deductively, this is a consequence of the demand that rules of inference be read collectively rather than distributively, and semantically it is a consequence of the way in which non-standard model structures were constructed. So both the inferential paralysis by deductive insulation of the first horn of the dilemma and the inferential collapse into universal assertion of the second are avoided. The two respects of non-standardness which generate non-standard worlds are thus both natural expressions of principles one would want antecedently to apply to beliefs.

Non-standard worlds thus give us the means to represent a whole spectrum of varieties of belief for each particular claim *p* which might in some sense be believed. We can, for instance, represent *bare* belief that *p* by a world in which *p* holds but none of the conjunctions of *p* with other beliefs holds in that world. Although *p* would be assented to, it has not been absorbed and integrated into the beliefs of the individual. At the other end of the spectrum, *total* belief that *p* can be represented by a world in which *p* holds, and if *q* is any other thesis consistent with *p* which holds in the world, *p* & *q* holds. Because of the way we developed the semantics of non-standard worlds, this entails that the consequents of any conditionals with *p* as their antecedents will hold in the world as well. In such a case, the belief that *p* has "sunk in," been considered in the contexts of all the other beliefs, and its consequences in those contexts rigorously adhered to. Along another dimension, we may consider the minimal belief in *p* as one where its negation is represented as also holding in the world, or, less drastically, where the negations of some conjunctions containing *p* hold in that world. Thus possible worlds induce a natural array of degrees of belief for each proposition, but these varieties are indexed by the non-standard truth tables involving *p* (rather than for instance, the real interval between 0 and 1). We believe this greater flexibility in the representation of the varieties of belief-attitudes one may have toward a single claim to be one of the prime virtues of the apparatus of non-standard possible worlds.

To be sure, using the apparatus of non-standard possible worlds to describe belief is not simply tantamount to the claim that the beliefs of an individual speaker at a time constitute a set of sentences in some language which is only weakly consistent. For the model of beliefs ought to be, not the set of sentences of some language, but a belief *world* or set of worlds, which *satisfy* such a set of sentences, that is, *the world which is just exactly as some individual believes it to be*. This is non-standard world, inconsistent and incomplete. The paradigmatic expression of a belief is a disposition to sincerely assent to a statement in some language, but this does not entail that all beliefs are adequately captured in a set of sentences in any particular language. We have argued above that the set of sentences of a language which are the expressions in that language of a set of beliefs will be only minimally consistent. This is not to say that there is not more to belief than this set of sentences. That component of one's beliefs which is not expressible in the particular language being considered is to be represented by the particularities of a non-standard world which, while it satisfies the minimally consistent and complete set of sentences of *L* which are the linguistic expression of those beliefs, also differs from other models of those sentences in various ways not specifiable in *L*. This stipulation amounts only to the recognition that there need be no unique language in which all of one's beliefs are expressible.

Let us represent the beliefs of an individual at a time with a non-standard possible world, his *belief*-world at that time. It was argued just above that this association need not be taken as implying that the individual whose beliefs are so represented could tell us everything about the world we use to represent his beliefs (even though this "everything" makes a weaker claim than might at first appear, since the world to be specified is itself incomplete). It is also worth pointing out that while the believer on this account believes in the existence of things we would describe as quasi-objects, with relations obtaining among them which conform to the principles expressed in our model theory (by means of a complicated structure of sub-relations into which the quasi-relation is partitioned), this does not imply there is anything in those beliefs which corresponds to the particular notational devices we have used in the metalanguage in which the model theory of non-standard theories is couched. In our treatment of non-standard worlds, for instance, we represented element-specifications in a notation which exhibits their compositional properties in a particularly perspicuous manner—each element-specification had the explicit form of a product by superposition and schematization of standard individuals. This notation is not expressible in the language L, whose weakly consistent theories are being modelled. One should therefore not require that a believer evince an ability explicitly to deal with element-specifications. We seek to represent beliefs by non-standard possible worlds, composed of quasi-objects and quasi-relations among them. The minimally consistent set of statements of *L* believed are true in this world. But the machinery of element-specifications and partitioned relations is *our* notation, a particular metalanguage we have adopted for discussing such worlds of quasi-objects. It is the worlds themselves.

not our metalinguistic representations of them in standard algebraic language, which are here to be taken as models of belief. Belief is a relation to such a world. That relation need not express perspicuously the potential decomposition into standard worlds which the notation in which the model theory was expressed was designed to make clear. In particular, the representation of belief by an associated non-standard belief-world is not to be taken as a claim that introspection on the part of the believer can recover a representation of that world such as we have employed, from which an account of its decomposition into standard worlds is immediate. It is not a trivial task for a believer even to be able to represent the (minimally consistent set of) sentences of *L* which he believes as the product in some (not unique) fashion of the union and intersection of complete and consistent theories of *L*. How much more difficult, then, it is to represent the facets of one's beliefs which are *not* expressible in *L* (indicated by a selection among the non-standard worlds satisfying the right sentences of *L*) by the schematization and superposition of *objects*. This last sort of project is internal to the theory we are presenting of non-standard worlds as representatives of beliefs, and is a task of the theoretician, not a capacity attributed to believers simply in virtue of being such.

So belief is to be taken as a relation a believer has at a time to a non-standard world. That world must satisfy all and only the statements of a language *L* which the individual believes. But one can believe more than can be said in any particular language, and this unverbalizable surplus is represented by the selection of a particular non-standard world from all of those satisfying the right sentences. The believer is not presumed to have the conceptual means for expressing this selection explicitly (everything so expressible gets packed into the sentences of a suitably powerful language *L*).

In the next section we will consider some aspects of *inquiry* (the methodical transformation of belief) which become apparent when one considers belief according to this model. We will not here address the sticky and interesting question of how linguistic, behavioral, and other kinds of evidence can justify the attribution of particular beliefs (the assertion of a relation to a particular world). We have advanced only general reasons to conclude that this issue will be more tractable once non-standard worlds and incomplete and only weakly consistent sets of sentences are admitted as legitimate tools.

SECTION 20

Methodological Realism and the Convergence of Inquiry

The project which will occupy the rest of our essay is that of giving an account of the ontological status of non-standard worlds as compared to standard, consistent and complete possible worlds. The apparatus of non-standard possible worlds as we have developed it so far is invaluable in the perspicuous presentation of an account of the various senses of ideality and reality whose claims must be adjudicated in order to settle the issue of ontological status. Accordingly, we will use that apparatus to define the notion of an *inquiry* which seeks an accurate and complete representation of things as they really are, and to define various constraints on the methodologies governing such inquiries as they relate to the reality which is presumed to be their subject-matter. Thus equipped, we will appeal to the example of the metaphysical views of Peirce as a concrete historical reference-point for a canvassing of alternative metaphysical views couched in the language of non-standard possible worlds.

Inquiry is the rationally controlled transformation of belief over time. The classical philosophical tradition has focussed its attempts at epistemological explication of this process on two particularly prominent motive forces which help drive inquiries in general. First, an *empirical* component of inquiry may be discerned: one constantly acquires new beliefs and discards old ones as the result of observations made in accord with widely shared non-inferential reporting practices. Second, a *rational* component of inquiry can be distinguished. Within a specified observational situation, one's beliefs are transformed according to complicated canons of inferential coherence, which dictate the practical incompatibility of some set of previously held beliefs, or require that such a set be amplified by the addition of a belief it entails (which entailment had been hitherto unnoticed). The observational and inferential practices which control the transformation of belief in this way may be referred to collectively as the *cognitive methodology* of the believer.

The stratagem of representing the beliefs of a speaker at a time by a non-standard possible world which satisfies all and only those (in general only weakly consistent) beliefs allows the exhibition of any particular inquiry as a *path* from one non-standard world to another, on the complete lattice of all such worlds. When a measurement provides a firm figure for the charge of a new particle under investigation, for example, the inquirer moves from a belief-world in which the quasi-objects which are the particles as they are believed to be are relatively more indeterminate to one in which they are relatively less indeterminate. Prior to the experiment, the inquirer's belief-world may conceivably have satisfied mutually inconsistent claims about the charge of the particle in question, or it may have satisfied no claims at all regarding that charge (i.e. neither was it true in the initial belief-world that the particle had a determinate charge, nor that it did not). The experimental measurement, however, moves the inquirer to a new belief-world. Of course, the belief-world arrived at in this fashion may still satisfy mutually inconsistent claims about the particle, and may remain incomplete in other regards.

In the sorts of cases which come most readily to mind, change of belief may be represented by a move from a world which satisfies a set S of sentences to a world which satisfies a different set S' of sentences, relative to some fixed language. As we have pointed out, however, some changes of belief may be better represented by movement from one non-standard world to another *within* the equivalence class of worlds which satisfy the same set of sentences in that language. Thus, someone may come to believe that a clarinet sounds like *this* and not like *that*, and not have sufficient musical vocabulary to express that alteration of belief linguistically (though the change of belief may have behavioral consequences for discriminations the believer would be disposed to make). The acquisition of such a new belief ought then to be represented as a movement from a belief-world in which clarinets have exceedingly indeterminate sounds (clarinets are quasi-objects in such worlds, their phonic properties incomplete and perhaps inconsistent) to a world in which they have a more determinate sound-range. The sentences of an official language L which are satisfied by these two worlds may be identical. The belief-worlds differ only in the way clarinets sound in them, and that difference need not be expressible in L.

A cognitive methodology with empirical and rational components must determine for each belief-world a set of admissible transformations by inquiry. We may represent such a methodology by an accessibility relation between possible belief-worlds, which will in turn determine a set of admissible paths through the lattice of non-standard possible worlds, corresponding to methodologically sanctioned inquiries. Our interest will not be in the details of the constraints which methodologies so codified impose on inquiries which recognize their authority, but concerns rather conditions which it is reasonable to insist that such methodologies must meet if we are to consider them candidate ways of seeking to find out how things really are in the first place. For it is clear that not all disciplined transformation of belief need have the representation of reality as its goal. A monastic community might take as the goal of its discipline the transformation of belief so as systematically to encourage or enforce the development of habits of action taken to be desirable from a religious or moral point of view, caring nothing for the *correctness* of the views thus arrived at, in the *cognitive* sense of that term. The question we wish to address is how to distinguish methodologies whose aim is the cognitive one of discovering how things really are. The task of offering conditions sufficient for such discrimination lies outside the scope of our enterprise. Instead, we will seek to elucidate one popular and plausible contender for the status of *necessary* condition for such discrimination, a view which can be discussed more precisely in terms of non-standard possible worlds.

The view in question may be called "methodological realism", enunciated well by Peirce:

> There may be some questions concerning which the pendulum of opinion would never cease to oscillate, however favorable circumstances may be. But if so, these questions are *ipso facto* not *real* questions, that is to say, are questions to which there is no true answer to be given. (CP 5.461)

Methodological realism requires two things of cognitive inquiries. First, that every question which is admitted as a proper subject of inquiry have some answer, and, second, that the aim of cognitive inquiries be to discover such answers. Thus formulated, the doctrine can admit (as Peirce would not) that there may be questions with true answers which cannot be established by inquiry. Methodological realism describes the *goal* of cognitive inquiry as the settlement of opinion by the determination of the truth. There are two issues here, what sort of methodological constraint expresses the *aim* of inquiry to bring our beliefs to some conclusion, and how "truth" or "how things really are" is to be interpreted as related to the ideal endpoints aimed at by cognitive inquiries. The second issue (including the question of the status of truths undiscoverable by inquiry) will be dealt with below when we discuss the thesis of *ontological* realism, contrasting it with Peirce's ontological idealism. The first issue Peirce approached through the notion of the methodological importance of the *convergence* of inquiry.

We will follow Peirce's suggestion, and express methodological realism by means of a distinction between inquiries, (represented by paths through the lattice of non-standard belief-worlds), which converge and those which diverge. We will not consider Peirce's account in its details since, as Quine has pointed out, it suffers from

> ... a faulty use of numerical analogy in speaking of the limit of theories, since the notion of a limit depends on that of 'nearer than', which is defined for numbers and not for theories.[87]

Using the lattice-structure which relates non-standard worlds, on the contrary, will enable us to define a precise sense of "limit" and "nearer than" which applies to the belief-worlds which make up inquiries. Methodological realism will then be the claim that lattice-convergence is a regulative ideal of cognitive inquiries. Our apparatus will enable us to distinguish a number of different senses in which this ideal could be taken to constrain particular methodologies.

We have used non-standard possible worlds to represent beliefs, and sequences of them to represent inquiries. We have shown above how the set of non-standard possible worlds can be treated as a complete lattice generated by the operations of superposition and schematization. The notion of convergence (and hence of limit) is defined primarily for topologies, numerical convergence being a special case, deriving from a particular order topology on the natural numbers. Mathematicians have shown how to associate topologies with suitably well-behaved lattices. In Appendix V on Convergence, we show how the intrinsic order topology of the complete lattice of non-standard possible worlds can be used to define convergence of nets (indexed by upper-directed paths), of which sequences or paths are a special case. Intuitively, the sense of "nearness" which is employed in this definition of convergence is nearness in respect of non-standard composition out of standard worlds. It is with respect to the way in which non-standard worlds are constructed by the superposition and schematization of standard and non-standard worlds that a sequence of such worlds may be said to converge. We have *not* imposed any kind of similarity measure on standard, consistent and complete worlds. Our approach does not permit us to compare such standard worlds with one another for "nearness", nor to talk about a convergent sequence of them (except eventually constant ones). For the structure we use to define convergence is the *lattice* structure, which expresses only how a world may be resolved algebraically as the product by superposition and schematization of a set of standard worlds.

To see what is at stake here, it is best to look at a *resolution convergent* inquiry. Such an inquiry is a set of worlds, indexed over time, such that each successive member of the sequence is a component of the preceding world (i.e., is a world which was superposed or schematized with some other to yield the preceding term of the sequence). Such a sequence will always be of finite length, and will terminate in a standard world to which it converges (somewhat trivially, because of its finite length) by our criterion. This process of resolution proceeds by sorting out and untangling the various components of an initial belief-state. The result is a set of beliefs linguistically expressible by a consistent and complete theory, a set of standard beliefs which can be conceived of as having been observed and confused with alternative standard belief worlds, resulting in an overall, non-standard belief world. The process of resolution is thus a certain kind of clarification of one's views' One understands one's own views better when those views can be exhibited in a precise way as a certain kind of product, by superposition and schematization, of standard belief worlds.

Of course, there are many other exigencies of clarity and cogency to which inquiry is subject. Resolution of one's views in this sense is not an over-riding imperative. It may be methodologically more appropriate at a certain stage in an inquiry to seek more empirical information, for instance, than to try to untangle one's present views to see just how one's current belief world is constructed. We shall have more to say on this issue below when we discuss methodologies from the standpoint of our lattice structure. The introduction of this notion of a resolution sequence is meant to give some indication of the kind of convergence according to non-standard construction by superposition and schematization we have captured in our technical formulation. It is insofar as the belief-worlds are inconsistent and incomplete that sequences of them converge on our lattice. It is nearness in point of non-standard construction out of standard worlds which is utilized in the definition. This is not the sort of convergence of sequences of worlds which other methodologists have considered, for the simple reason that they did not consider non-standard worlds at all. Our discussion below will accordingly deal with those aspects of the methodology of inquiries which concern the non-standardness of belief-worlds. The structures we define do *not* (and are not meant to) provide an account of what "convergence" might mean for a sequence of consistent and complete scientific theories, for instance.

The Appendix on convergence defines a precise sense in which inquiries as we have represented them can be said to converge on a settled opinion or way things have been found to be. and we have interpreted methodological realism as the claim that such convergence is a regulative ideal of cognitive inquiry. We have not yet said how this ideal might be expressed in constraints on methodologies which govern inquiries. Before we address such questions as whether it is reasonable to require that every methodologically admissible inquiry converge to some world, it will be useful to consider some more specific conditions which can be formulated in terms of our notion of convergence of inquiry. One generally laudable characteristic of an inquiry is that over the long run it converges to a *consistent* world. That is, we have a bias towards wanting our inquiries to be progressive in the sense of weeding out inconsistent beliefs. Of course it is the import of this whole study that such a requirement ought *not* to be taken as an absolute methodological requirement, but as a nice property, desirable when other things are equal. Our specification of the semantics of inconsistency is meant to show that in various circumstances the general desirability of consistency may be over-ridden by other methodological desiderata. A strategic bias toward consistency need constrain our methodological tactics less than has been previously thought.

The dual notion of *completeness* convergence concerns not the deductively motivated preference for consistency, but the inductively motivated search for completeness in our beliefs. This motive is, it would seem, less urgent than that of consistency. The push for a unified and universal science is not methodologically negligible, but not as urgent as the push for consistency where it is obtainable. That requirement is even more obviously merely a desideratum, to be carefully weighed against competing methodologic considerations. Other things being equal, it is a greater criticism of a theory that it is not consistent (though as we have shown, this is not an insurmountable objection) than it is that it is not complete. Completeness convergence is accordingly a less important methodological recommendation than consistency convergence.

The conjunction of these two conditions is the requirement that an inquiry converge to a standard, consistent *and* complete world. *Standardness convergence* in this sense is a very strong condition to put on one's methodology. To see what sort of a condition it is, consider methodologies and requirements on them somewhat more generally. A methodology is a specification of a set of *admissible* inquiries. One way of doing this is to express the constraint of admissibility by transitive and reflexive accessibility relations on non-standard belief worlds. A sequence of worlds with an initial element, that is, an inquiry, will then be admissible according to the methodology captured in the accessibility relations just in case each element of the chain is accessible from every earlier element. One may, of course, simply specify a set of admissible chains directly, simply as a set of such chains. Any characteristic shared by all inquiries which are admissible according to a particular methodology is *methodologically enforced* or guaranteed.

One such interesting condition is *universal convergence*. This is the requirement that every admissible inquiry converge to some belief world. The methodologies which meet this condition *ensure* that no divergent inquiry will be methodologically acceptable. It is not a trivial task to put conditions on an accessibility relation between belief worlds in such a way as to guarantee convergence in a non-trivial fashion. This condition can be strengthened by combining it with the other conditions we have considered. Thus *universal consistency convergence* will stipulate that each admissible inquiry converges to a consistent world. *Universal completeness convergence* and *universal standard convergence* conditions may be constructed similarly. Strengthening along an independent dimension, we can formulate *unique universal convergence*. This is the requirement that all admissible methodologies converge to the *same* world, although different paths to that goal are allowed. This condition may obviously also be combined with those of consistency, completeness and therefore standardness.

Each of these possible methodological requirements stipulates *a priori* an important characteristic of all inquiries admissible according to a methodology imposing the conditions. The conditions described in the last paragraph all entail the Universal Convergence condition. This means that they all remove from the realm of empirical possibility otherwise methodologically acceptable inquiries which do not converge. We may ask whether this is a question which is appropriately addressed at the level of *a priori* methodology.

SECTION 21

Peirce and Empirical Convergence

Peirce was inclined to insist that every admissible inquiry be such that it converges. In a passage which combines what we will call below Peirce's ontological idealism with this insistence he says:

> . . . to assert that there are external things which can be known only as exerting a power on our sense, is nothing different from asserting that there is a general *drift* in the history of human thought which will lead it to one general agreement, one catholic consent. (CP 8.12)

We are currently interested in reading the asserted identity in the other direction, as claiming that to be finding out about the external reality Peirce invokes one must drift toward that general agreement. Peirce's views will be considered in more detail below, but we see here that he is willing to insist on convergence to a single belief world as a criterion of an inquiry which claims to address itself to an empirical reality.

A number of questions should be asked about this view. First, even if it is accepted, does it rule out as empirically significant a methodology which does not *guarantee* convergence *a priori*, but merely aspires to it? If a methodology comprises some convergence a *sine qua non* of admissibility, then whether a particular inquiry converges or diverges is an *empirical* matter. One just has to wait and see, engaging in actual inquiries, transforming beliefs according to the methodology, moving from one belief-world to another. Luck may be required to make the transformations of belief which will lead to convergence. Many admissible or obligatory transformations of beliefs may not contribute to convergence at all except in fortunate, empirically determined circumstances. Empirical inquiries must aspire to convergence. But actual inquiries we class as empirical do not obviously *guarantee* convergence, and there are horrible, unempirical methodologies which do. Thus, the method of Tenacity[88] which consists in maintaining one's current beliefs come what may, or various more socially totalitarian versions of this superhuman dogmatism would satisfy the Universal Convergence condition without being empirically praiseworthy thereby. Is it not imaginable, for instance, that a perfectly acceptable methodology would have the characteristic that any admissible inquiries which began with a belief world in a certain region of the lattice (or one with some other property) would converge, while those initiated elsewhere would not? This would be a methodology with an empirical presupposition to its "guarantee" of convergence. Perhaps even our own scientific methodology has this characteristic. Is it clear that current ideas of scientific methodology, if applied to, say, Parmenides' belief-world, would result in the sort of empirical-technical convergence we fondly believe ourselves to have assured beginning with Newton's world? Methodologically *guaranteed* convergence may exhibit only the advantages of theft over toil usually associated with a priorism.

Another question we must ask is whether empirical methodology must seek to ensure *unique* convergence. Peirce says:

> Different minds may set out with the most antagonistic views, but the progress of investigation carries them by a force outside of themselves to one and the same conclusion. This activity of thought by which we are carried, not where we wish, but to a fore-ordained goal, is like the operation of destiny. No modification of the point of view taken, no selection of other facts for study, no natural bent of mind even, can enable a man to escape the pre-destinate opinion. This great hope [in first draft: "law"] is embodied in the conception of truth and reality. The opinion which is fated to be ultimately agreed to by all who investigate is what we mean by the truth, and the object represented in this opinion is the real.(5.407)

Peirce is unambiguously ruling out a methodology according to which, for instance, some inquiries converged to one belief world and the others converged to another, or diverged. The difficult status of this claim is indicated by Peirce's shift over the years in the precise official formulation of it which he endorsed. In the early days of "The Fixation of Belief", he thought of this unique convergence as a *law*, constitutive of empirical inquiry and reality alike. In later years, however, he treats unique convergence as a merely regulative ideal. Unique universal convergence is a state empirical inquirers *aspire* to, an outcome they *hope* and even *believe* will apply to their efforts, but an issue which is essentially empirical and undemonstrable *a priori*. In short, he came to think of unique universal convergence as something which could be *guaranteed* in advance only by such ferociously anti-empirical methodologies as that of Authority ("Everyone believe what *S* now believes!") Such a position is entirely compatible with concurrent belief in a real world that is independent of our actual beliefs and to which those beliefs should aspire to be faithful, that is, is equivalent to belief in the empirical convergence of appropriate inquiries.

These considerations suggest a weakening of Peirce's condition.

If we deny Universal Convergence, *conditional unique universal convergence* may still hold. This condition requires of every admissible inquiry that *if* it converges, then it converges to the same world that every other admissible convergent inquiry of that methodology has as its limit. In such a methodology, it is an empirical matter whether any admissible inquiry will converge or not, but there is only one possible admissible outcome for such convergence if it does occur. The other strengthenings of Universal Convergence which we considered can also be expressed in weaker, conditional forms. Thus *conditional consistency convergence* states that every convergent admissible inquiry in a methodology converges to a consistent world, *conditional completeness convergences* states that every convergent admissible inquiry converges to a complete world, and *conditional standardness convergence* is the conjunction of these two conditions. The sort of methodological thesis exemplified by these various conditions is characteristic of a certain kind of *realism*, which we will consider further below. The leading intuition is first that "good" cognitive inquiries ought to lead us to find out something (settle our beliefs, represented here by convergence, not just of the sets of sentences held true, but of everything believed, convergence of sequence of *worlds*). Second, good inquiries ought to lead us to believe in the *right* world, the one which is real, actual, or distinguished in some other inquiry-driving fashion. One's prejudices about the nature of that world will accordingly color the methodological proscriptions concerning admissible inquiries one is prepared to subscribe to. In particular, if one is in some fashion convinced *a priori* that the world our empirical inquiries strive to be adequate to just *must* be a consistent one, then conditional consistency convergence is a reasonable methodological constraint to impose on inquiries, and one would look next at the various detailed ways of formulating this requirement in terms of practically describably accessibility relations from one non-standard world to another (e.g. stipulating that no world *w*′ which is inconsistent in some respect in which world *w* is consistent be methodologically accessible from *w*, and so on). Our current interests are more abstract than this project, however. We are considering various kinds of methodological convergence on the lattice of non-standard possible worlds in order to elucidate the ontological and epistemological status of non-standard possible worlds.

One worry which our use of the notion of convergence might raise is the following. It might seem that treating various kinds of convergence as methodologically meritorious is a concession to *conservatism*, which by ruling conceptual revolutions out of court at the outset seeks to guarantee that an inquiry will not arrive at conclusions radically different from those beliefs which occur early in the inquiry. If this objection were correct, it would be a serious criticism of our account. But in fact the global constraint of convergence does *not* rule out locally discontinuous, radical, or revolutionary changes of belief. Specifically, in our mathematical model (as in the special case of numerical sequences) convergence does not mean that if $m < n$ that the $n+1^{st}$ element must be "closer to" the n^{th} element than the $m+1^{st}$ is to the m^{th} (where the lattice ordering determines the sense of "nearer"). Nor is there any pair of worlds w_1, w_2 (or, again, numbers, in the special case) which are too "far apart" for w_1 to be the n^{th} and w_2 the $n+1^{st}$ elements of a convergent sequence of worlds (numbers). This is clear from the consideration of the trivial case of a sequence which is constant after a certain point. So there is no change of belief which is too radical or revolutionary to be contained in a convergent inquiry, and convergence is not the counsel of conservatism. More precisely, a methodology may br called *conservative* insofar as it attributes a special positive weight, in its evaluation of possible transformations of belief, to *retaining* beliefs currently held, or which have remained unchanged over the recent course of inquiry. Conservatism thus described is a constraint on methodologies at the same level as our various convergence conditions, and the present point is just that it is a constraint which is completely independent of considerations of convergence.

Having pointed this out, however, it might seem that we have opened the door to a more serious criticism. If the requirement of global convergence excludes *nothing* at the local level (forbids no particular transition), then isn't it a *vacuous* constraint on methodology, which after all can only operate at the local level, governing individual changes of belief via an accessibility relation? The requirement is *not* empty, as we see when we distinguish the *two* levels of constraint which are relevant here. First, we have constraint of an inquiry (path) by a methodology (accessibility relation determining admissible paths). Second, we have the constraint of methodologies by meta-theoretic principles like unique universal convergence. That permission for any particular transition at the first level is consistent with any particular prohibition (at least of the well-behaved ones we have considered) at the second level does *not* mean that second-level constraints *on methodologies* aren't genuine. To be genuine is to exclude *some* methodologies, partition the infinite space of possible methodological accessibility re.ations into those which permit and those which do not permit paths prohibited by a meta-level principle (such as the requirement that all pairs of paths with a common element converge to the same world). In fact a requirement such as that just stated clearly *is* a significant constraint on methodologies. The *apparent* vacuity of these meta-level principles arises only from the confusion of the two different sorts of constraint on inquiry which are at issue here.

Finally, we may consider briefly the complementary objection, that our talk of convergence of cognitive inquiries is utopian and hopelessly optimistic, since actual inquiries cannot be expected to conform to such an expectation. (Such a view might be motivated either by a low opinion of current cognitive practice or contemporary methodological sophistication, or by pessimism about the influence of the latter on the former.) To this we respond that methodology is primarily a matter of how we *appraise* inquiries, and such appraisals may be related to the actual course of previous inquiries only in historically and sociologically complex ways. Future inquiries can be affected by our methodologies, but only via the appraisals of possible course of inquiry which we make according to *them*. The objection would have a point were we making specific methodological recommendations, but is without force at the level of meta-theoretic conditions which might be imposed on methodologies, which, after all, is where our concerns lie.

SECTION 22

Inquiry and the Social Point of View

One further feature of inquiry which we should show how to represent using this apparatus before passing to the explicit consideration of the ontological question is that of the social or communal nature of inquiry. Thus far inquiry has been considered as an essentially *individual* affair. And inquiry has been identified with a sequence of belief-worlds which satisfy exactly the beliefs of an individual at a time. Each such world includes a past and a future as well as a present (so we must distinguish the historical sequence of believed-events within the world which represents a speaker's beliefs at a time from the sequence of belief-worlds which constitutes an inquiry over time). But inquiries within a particular community (e.g., metallurgists) is not a matter of individually independent inquiries carried on by isolated investigators, striving for goals which may or may not be shared by their colleagues. The transformation of the beliefs of one inquirer is typically dependent upon the current beliefs of other inquirers who are recognized as engaged in related projects. Our purposes do not require that we go into the matter of representing this social dimension of inquiry in much detail (a substantial undertaking in its own right), but it is worth indicating how such a representation might be accommodated. An obvious strategy is to use the two modes of world fusion we have been unfolding all along to combine the beliefs of the community at a time into a single non-standard belief world. Thus we can represent the union of all the beliefs of all the inquirers in a given community at a particular time just by the superposition of their individual belief-worlds. Similarly, we can represent the intersection of all of the communal beliefs at a time by the schematization of all of the individual belief-worlds. These constructions are just non-standard possible worlds like any others, so the various notions of convergence considered above can be defined for these product worlds as well. Exactly what gets represented in this fashion?

The communal superposition belief-world will contain a great many objects and relations on them which not very many of the members of the community believe in. It will undoubtedly be an inconsistent world, with inconsistencies representing disagreements between members of the community. On the other hand, the communal schematization world will omit a great many objects and relations on them which most of the members of the community do believe in. It will be an incomplete world, with respects of incompleteness representing disagreements between members of the community. Greater agreement in the individually held beliefs represented in the communal superposition and schematization belief-worlds is thus a matter of convergence towards relatively more consistent worlds on the part of the sequence of superposed belief-worlds, and of convergence toward relatively more complete worlds on the part of the schematized belief-worlds. Just as the settlement of opinion represented by convergence on the lattice of belief-worlds is a plausible criterion of cognitive significance of an individual inquiry, so the approach to are *consensus* represented by convergence of superposed and schematized communal belief-worlds is a plausible criterion of cognitively successful *social* inquiry.

This is not to say that convergence of superposed or schematized communal belief-worlds cannot be methodologically engineered in cognitively perverse ways. A methodology which ensured that there would always be radical dissenters in the community, who disagree with nearly everything held by the majority might fill up the communal superposition belief-world in such a way that the sequence of such worlds converges to a *maximally* inconsistent world. Such perverse convergence can easily be brought about by methodological stipulation (e.g. that a world *w*′ is methodologically accessible from w only if there is some state of affairs which holds in w while its denial does not, which holds as well as its denial in *w*′, and so on) and are clearly not cognitively praiseworthy. Similarly, a methodologically assured radical dissenter will have the effect of making the schematized communal belief-worlds converge toward a maximally incomplete world in which no non-logical statements are true, a tendency which is again not indicative of the success of an inquiry. Thus the convergence of the sequence of schematized communal belief-worlds and the convergence of the sequence of superposed communal belief-worlds does *not* ensure increasing agreement among the members of the communities whose beliefs are considered. Sequences can converge in different directions on the lattice of non-standard possible worlds. Notice however that where such convergence is methodologically engineered in a *perverse* way, so that it represents increasing *dis*agreement instead of increasing agreement, the sequence of schematized worlds and the sequence of superposed ones converge toward *opposite* ends of the lattice, one becoming less consistent, the other less complete.

It is natural to suggest, then, that we represent increasing communal consensus over the course of inquiry *not* by the convergence of the communal superposition belief-worlds and communal schematization belief-worlds individually, but by their *dual* convergence *to the same world*. We have sought to represent the settlement of individual opinion by convergence of belief-worlds on the lattice of all non-standard possible worlds. The current suggestion is that communal agreement (the settlement of opinion along the *social* dimension) is represented at a time by the world which is the superposition of all the individual belief-worlds of members of that community at that time and by the world which is their schematization. The non-standard structure of these worlds represents degrees and respects of agreement and disagreement within the community, in virtue of the way what is true in those schematized and superposed worlds is related to what is true in the (themselves non-standard) individual belief-worlds (according to theorems T1 and T2). Since this non-standard structure which represents agreement and disagreement among the views fused by schematization and superposition is just the structure with respect to which we have defined various kinds of convergence, increasing communal agreement is representable by such convergence. The problem of direction of convergence, so as to represent increasing agreement only, and not increasing disagreement, is readily solved by requiring that both the superposition sequence of communal belief-worlds and the schematization sequence of those worlds not only converge, but converge to the same limit-world.

A few remarks about this suggestion are in order. First, the various methodological constraints which might be imposed on inquiries in terms of convergence conditions which were discussed above are all compatible with a move from the notion of individual inquiry convergence to the representation of the social dimension by double convergence of the sequences of superposed and schematized belief-worlds to the same limit. Thus we can require of every admissible *communal* inquiry that it double converge (the social version of Universal Convergence), that not all need double converge, but if they do they must all double converge to the same world (double convergence itself only takes place, by definition, if the sequence of schematized worlds and the sequence superposed ones converge to the same world, but two double convergent communal inquiries might double converge to different worlds, unless a methodological restriction is applied), etc.

Next, the technique of using double convergence to represent increasing communal agreement need not limit us to describing internally structureless communities such as our simple discussion above considered. The sociological aspects of cognitive inquiry are ill-understood, and we should expect that a better understanding will involve attributing all sorts of social complexities to a community of inquirers, distinguishing sub-communities of experts, in-groups and out-groups, theoreticians and experimenters, and so on. If we had some such detailed account of the social functioning of inquiring communities, we would of course seek to represent the views of that community at a time by some more complicated structure than the pair of the schematization of the belief-worlds of all the members of the community and their superposition.[89] Nonetheless, the use of double convergence to represent increasing agreement within a simple community will be applicable to any sub-communities analysis might provide, and this representation of agreement by non-standard structure might well be crucial to describing the detailed interactions of such sub-communities over time. At any rate, we mean only to suggest a principle, double convergence, by means of which communal agreement may be represented by the non-standard structure of belief worlds, and by convergence on the lattice of such worlds.[90]

SECTION 23

Realism and Idealism

The apparatus we have developed for the treatment of certain aspects of inquiry and cognitive methodology is subject to two different sorts of ontological interpretation. Considering a single inquiry which converges to a standard consistent and complete world, one may take either the limit world or the converging belief-worlds to be ontologically primary. The (ontologically) *realistic* interpretation of such an inquiry takes the limit world to which the inquiry converges to be the real world, the world which the inquirers inhabit, and which controls the inquiry throughout. The belief-worlds, inconsistent and schematically incomplete, which constitute the inquiry itself, are considered to be merely *ideal*, representations not of an ontic state of affairs, but merely of an epistemic attitude toward the limit world which is the object of the inquiry. The merely ideal status of the belief-worlds is exhibited by their failure to fulfill even the minimal condition of consistency which has traditionally been required for actual or possible existence. A proponent of such a position might accept non-standard possible worlds as an acceptable formal language for the expression or representation of *beliefs*, but will not accept them as appropriate representations of any actual or possible states of affairs. Our attitudes, themselves merely ideal objects dependent upon the subject whose attitudes they are, are admitted to be inconsistent and incomplete. It is denied that anything real, independent of the attitudes anyone might adopt toward it[91] can be inconsistent or incomplete.

Opposed to this interpretation of the convergent inquiry in question is an *idealistic* interpretation. According to this view, the worlds we actually inhabit and should think of ourselves as inhabiting are the belief-worlds of each stage of the inquiry. They alone are to be conceived of as real and actual. In virtue of a certain sort of structure which the sequence of such belief-worlds has (its resolution-convergence properties, as botanized in the previous section) the inquirers can—if they so choose—project an *ideal* limit to the actually infinite process of transforming actual current beliefs into new beliefs. They live out their lives *in* the inconsistent and schematic worlds of their beliefs. The social and historical order of inquiry, represented paradigmatically by double convergence of communal beliefs, permits the inquirers (or someone else who studies them) to project an ideal limit toward which their actual transformations of belief are tending, and which the inquiry can accordingly be understood as about, as aiming towards, as seeking to be adequate to. The limit-world, according to this interpretation, is an epistemological *construct* from an objectively existent inquiry.

Notice that according to either of these mirror-image interpretations of inquiry, the relation between the real and the merely ideal, between the elements of ontology and the constructs of epistemology, is explicated in terms of the convergence of inquiry. For the idealist such as Peirce, the real is simply *defined* as the limit to which an appropriately disciplined inquiry converges (cf. 5.407 quoted above, for instance). For the realist, on the other hand, Peirce's definition should be stood on its head and made to read "an appropriately disciplined inquiry (that is, one which is admissible according to a *correct* methodology) is one which converges to the real world." Putnam[92] and Sellars[93] are realists in this sense. This correlation of the notion of reality with the notion of convergent methodologically admissible inquiry is shared by the ontological realist and the ontological idealist in the interpretations sketched above. It is itself a substantive philosophical thesis, one which is denied, for instance, by Paul Feyerabend.[94] To consider convergence of inquiry as a *sine qua non* of cognitive success, that is, to take one's inquiry as trying to find out about some reality, is often taken, by Peirce and others, to be itself a form of realism.

This brief indication of some of the varieties of convergent conditions which could be imposed on methodologically admissible inquiries points to some of the different more detailed forms such a realism might take. Realism in *this* sense, taking one's inquiry as *about* something in the sense given by some kind of convergence condition, is not our present concern. To distinguish it from the ontological realism and idealism we are considering, call it *methodological* realism (in virtue of its essential use of convergence conditions on methodologies). The different ontological interpretations of convergent inquiries sketched above are both methodologically realistic.

SECTION 24

The Example of Peirce

It will be useful to consider a particular example of the general approaches we have presented abstractly in the terminology of idealism and realism. Accordingly, we shall, in this section, examine briefly the views of the early Peirce on methodology and ontology. This discussion serves a dual purpose. On the one hand, we will see several important aspects of Peirce's views which have seemed mysterious to those approaching his work with standardist prejudices, and which become clear and persuasive when viewed from the perspective of the apparatus of non-standard possible worlds. On the other hand, the consideration of Peirce's sophisticated views about convergence, inquiry, and reality provides a concrete setting in which to argue the thesis of the ontological parity of standard and non-standard possible worlds which is our ultimate aim.

Peirce is a methodological realist, but an ontological idealist. One of his more important legacies to later pragmatists (taken up particularly by Dewey) was his doctrine that the objects of knowledge inhabiting the limit-world approached by a convergent methodologically admissible inquiry are the intelligible *products* of such disciplined inquiry, not its ineffable *causes*.[95] Peirce's Kantianism consists precisely in this claim that the methodologically real is ontologically ideal.[96] This combination of views is often obscure, since Peirce is not careful to distinguish the two senses in which "real" is used. Because it is precisely in the interface between the methodological realism and the ontological idealism that Peirce's views about inconsistency and incompleteness lie, it is worth looking more closely at this region.

Peirce does not talk about inconsistency and incompleteness in those terms, but in terms of *vagueness* and *generality*. The following passage is characteristic:

> Perhaps a more scientific pair of definitions would be that anything is *general* in so far as the principle of excluded middle does not apply to it and is *vague* in so far as the principle of contradiction does not apply to it. (5.448)

Peirce called anything which is vague or general a *sign*, and claimed that ideas were signs in this sense.[97] His ontological idealism thus consisted of the claim that the signs in this sense which make up our beliefs at a time are what actually exist. In fact, Peirce has a more radical view still. He identifies a human being with his beliefs and the transformation of them over time which is inquiry.

> . . . the fact that every thought is a sign, taken in conjunction with the fact that life is a train of thought, proves that man is a sign . . . (5.314)

Man is thus the inhabitant of an inconsistent and incomplete world, a world of *signs* in Peirce's special usage. Since signs are coordinate with ideas, this view was entitled, Peirce felt, to be called an idealism. But idealism in this sense is perfectly consistent with methodological realism, the view that the controlled settlement of opinion (to which we have given precise expression in terms of the resolution convergence of methodologically admissible inquiries) is the criterion of truth. Thus Peirce wrote:

> When we busy ourselves to find the answer to a question, we are going upon the hope that there is an answer, which can be called *the* answer, that is, the final answer. It may be that there is none. If any profound and learned member of the German Shakespearian Society were to start the inquiry how long since Polonius had had his hair cut at the time of his death, perhaps the only reply that could be made would be that Polonius was nothing but a creature of Shakespeare's brain, and that Shakespeare never thought of the point raised. Now it is certainly conceivable that this world which we call the real world is not perfectly real but that there are things similarly indeterminate. We cannot be sure that it is not so. In reference, however, to the particular question which we at any time have in hand, we hope there is an answer, or something pretty close to an answer, which sufficient inquiry will compel us to accept. (4.61)

Here Peirce is envisaging methodological convergence to a non-standard world. Not only is every stage of inquiry a world of signs, fraught with vagueness and generality (that is, in his somewhat confusing usage, *ideals* or signs rather than *reals*) but so may well be the limit world which they seek to be adequate to, to find out about. The doctrine that this limit-world, consisting of the objects of knowledge *produced* by methodologically admissible convergent inquiry, contains vagues and generals (quasi-objects in our technical sense) is what Peirce calls Scholastic Realism. It is the view that inconsistent and incomplete objects are methodologically real.[98] Since man is such an object (or quasi-object) according to Peirce, this is also the question of whether *man* is methodologically real, i.e. whether he exists in the limit-world as an object of knowledge to which appropriate inquiries will converge.[99]

It is clear, then, that an ontological idealist like Peirce can tolerate the methodological reality of non-standard worlds and their constituents. If reality in this sense is ultimately *defined* by convergence of methodologically admissible inquiries, there seems no particularly compelling motive for requiring *standardness* convergence (i.e., convergence to a *standard* world). The ontological realist, on the other hand, does seem to have the intuition that the (methodologically and ontologically) real world just *must* be standard, that standardness is a necessary condition of actual existence, as opposed to the merely virtual status of inconsistent and incomplete objects. Peirce acknowledges such an intuition in the passage quoted above when he describes the situation in which inquiry converges to a non-standard world as one in which the (methodologically) real world is not perfectly real (i.e., standard). This is an intuition of the greatest importance for our overall project.

SECTION 25

A Critique of Standardism

Our purpose throughout has been to set out a framework to justify giving inconsistency and incompleteness an appropriate place within the arena of rational inquiry. The ideas we have considered are intended to show that inconsistency and incompleteness—and even the ambiguous identity relations of quasi-objects which such situations include—have a perfectly intelligible and rigorously formalizable structure. We have urged that non-standard possible worlds should be accorded exactly the same ontological status which the standard, consistent and complete worlds enjoy. (After all, from one point of view, those standard worlds are merely a special, degenerate case of non-standard worlds.) Thus we want to maintain the ontological *parity* of standard and non-standard worlds—the claim that *whatever* ontological status one accords to standard, consistent and complete possible worlds must be accorded as well to their non-standard brethren.[100]

In particular, we are concerned to raise the possibility that standardness convergence ought *not* to be taken as an absolute methodological constraint on all serious inquiry—that, on the contrary, inconsistency and incompleteness are tolerable and manageable in the progressive transformation of belief and practice which is inquiry. One should not in other words, consider convergence to a non-standard world as the hallmark of cognitive defeat, but rather must prepare for the possibility that *our* world—the real world, in which we live and breathe and have our being—is non-standard. The ontological realist and the ontological idealist differ, as we have seen, in their interpretation of which world *our* world is—whether it is the belief-world which represents the current stage of inquiry, or the limit-world to which that inquiry will (hopefully) converge eventually.

Our concern is with the relation of this dispute to ontological *standardism*, the view that all that exists, actually *or possibly*, is consistent and complete objects and the worlds comprising them. Ontological standardism depends upon maintaining a disparity of status, roughly, between beliefs and the things those beliefs are *about*. The former are allowed to be inconsistent and incomplete, the latter precluded from being so. More precisely, standardism claims that inconsistency and incompleteness are characteristics always of represen*tings*, and never of represen*teds*. Inconsistency and incompleteness are thus seen as intentional, subjective, linguistic, or epistemic characteristics, never as ontological ones.

To see how ontological standardism relates to idealism and realism, consider the various ways in which the represented/representing distinction central to it might be made out, and how plausible standardism is according to various such renderings. One natural way of making out the belief/object or representing/represented distinction in detail is to use the theory/model relation, a relation about which the semantic and meta-mathematical work of the last fifty years enables us to be fairly precise. On the one hand we have a set of sentences, a theory expressed in some formal language. On the other hand, we have a model structure, paradigmatically an algebraic one, which satisfies the theory according to canonical correlation of the syntactic structure of the sentences with the algebraic structure of the model. In using this idiom to discuss what the theory is about, it is usual to restrict attention to some distinguished model (or set thereof) out of all those that might satisfy the theory, called the *intended* model.

Will this interpretation of the representing/represented distinction give any aid and comfort to the ontological standardist? No, for we have shown how to construct non-standard models for inconsistent and incomplete theories. According to the model theory we have developed, *both* sides of the theory/model relations can be non-standard. Our beliefs, in whatever language they are expressed, are inconsistent and incomplete. We have shown how to model such beliefs by appealing to quasi-objects and their relations, and these objects and the worlds they constitute are non-standard in the extreme. We conclude that the construction of our non-standard model theory (which is semantically complete for *weakly* consistent—or inconsistent—first order theories) has removed any support the standardist might seek to derive from the theory/model relation for his claim that inconsistency and incompleteness are characteristics of representings only, and never of representeds.

Of course this argument is not final. The standardist can argue plausibly that the theory/model distinction does not adequately capture what is intended by the representing/represented distinction. After all, we say how non-standard models can be used to characterize *beliefs*, distinguishing them from any particular linguistic *expression* of those beliefs as a set of sentences. The consistent standardist can thus respond that non-standard models are representations of representings (namely beliefs), so that they do not conflict with his claim that only representings can be non-standard. The evidence of the theory/model distinction is not, to be sure, in favor of the standardist once a non-standard model theory is on the scene, but it cannot be taken as refuting him without begging the question at issue either.

What other rendering of the representing/represented distinction is available to capture the standardist's intent? One natural move to make is to appeal to a distinction between the order of *justification* and the order of *truth* by taking a belief to be about that to which it is responsible, to which it seeks to be adequate, that which is recognized in principle as having authority over the belief. What is represented is that which will determine the truth of one's beliefs. That truth-determiner, according to this version of the standardists claim, will be consistent and complete. The representings which are manipulated in more or less justified ways in the attempt to approach that truth may be and often are inconsistent and incomplete. This may be taken as just the surest evidence we could have of the distance they have to go to do justice to their objects. Seen from this point of view, the trouble with the suggestion that the theory/model distinction be employed is that non-standard models and weakly inconsistent theories are just two different ways of representing beliefs (according to the standardist) and hence can both be inconsistent and incomplete precisely by ignoring reference to what the beliefs seek to be adequate to.

But this version of the standardist claim will not do either. For we have shown how to represent the cognitive aspiration to truth, which methodologically controls inquiry, by resolution convergence of a sequence or net on the lattice of non-standard possible worlds. The methodological realism which the standardist argues from, the distinction between the (methodological) order of justification which governs the transformation of beliefs and the limit to which, if things go well, that inquiry converges as the truth is entirely indifferent to the question of whether or not the limit-world to which an inquiry converges is standard or non-standard. It might, of course, turn out that all the inquiries we independently consider methodologically admissible will be standardness-convergent, but merely appealing to the distinction between worlds which appear in the inquiry and the world to which that inquiry converges does not provide any evidence for the standardist's claim. Once again, since we have shown how to interpret the distinction between the order of justification and that of truth, or between inquiry and that which it aspires to represent adequately, in terms which do *not* require that the truth aspired to be consistent and complete, the standardist cannot non-circularly demonstrate his claim on the basis of this epistemological distinction. Justification can be represented by the local structure of the methodological accessibility relations which determine the admissibility of inquiries, while truth, that to which the inquiry seeks to be adequate and which, ideally, exercises authority over it, is identified with the limit-world, which may or may not be standard.

Still, such a counter-argument also cannot be considered decisive against the standardist, for other formulations of his claim are possible. One such possible reformulation seems particularly important in the light of current work on reference and representation. The standardist might argue that not just any "adequate to" relation defined by methodologically admissible convergence of an inquiry to a limit-world will do. Instead, this line continues, that limit-world must be the *cause* of the methodological convergence, in some special fashion which current research is endeavoring to determine in detail. It is not enough that an inquiry converge to a world, the world must cause that convergence *via* methodological constraints (again, not just any causation will do, what is envisaged is a special kind of causal influence, no doubt crucially involving the causal etiology of reports and the circumstances of introduction of various social linguistic practices). Only this special causal relation to its subject-matter distinguishes empirical inquiry from, say, the achievement of a literary consensus concerning the aesthetic values of some text. For an empirical inquiry might converge in a methodologically admissible fashion to a limit world which was *not* causally responsible for the course of that inquiry. Such an inquiry would on this view have gone wrong, and *ceteris paribus* the admissibility of inquiries of this sort ought to count against a methodology (though it may be that, given the way the world is, the best methodology *in principle* available to us is one for which it is an empirical matter whether convergence will be to the causally responsible world—i.e., some admissible inquiries will not so converge). This distinction between causally appropriate methodologies and those which are not is one we have not built into our discussion of inquiry in terms of the lattice of non-standard possible worlds, so it is a plausible place for the standardist to erect his fortifications.

Such causal considerations do seem to be the crux of the dispute between the ontological realist and the ontological idealist. That dispute as we have described it, boils down to the question of what the causal motors of inquiry are and by right ought to be. The ontological realist's claim that inquirers "inhabit" the limit-world of a methodologically correct inquiry (we are assuming methodological realism) can be glossed in terms of the causal interactions between that world and the inquirer and his beliefs. It is the things in that world which are causally responsible for the course inquiry actually takes. Even the claim that that limit world is, if things go well with the inquiry, the *actual* world can be expressed in terms of the causal relations. For "actual world" may be understood as an indexical expression like "*this* world",[101] and the referent of such an indexical-demonstrative is plausibly determined by consulting the causal interactions of the speaker with his surroundings.

The ontological idealist can also accept a gloss of his position in terms of the causal motors of inquiry, expressing his disagreement with the realist as a disagreement about the causal antecedents of the transformation of belief which is inquiry. The idealist contends that the causal motor of inquiry is the local transformation of belief according to methodological constraints which are themselves *part* of those beliefs which are transformed. (This could be expressed by taking one's beliefs at a time to be represented not just by a belief-world but by a belief-world together with its methodological accessibility relations.) For the ontological idealist, the only thing which can be causally responsible for a transformation of belief is previous belief. According to this idealistic interpretation, the realist's view invokes *final causation* of the process of inquiry by its goal. For the idealist the world an inquirer inhabits, the world which is causally efficacious for the transformation of inquiry, is his current belief-world. The sequence of those worlds are linked by efficient causation. Some of the transformations which are thus effected will be methodologically acceptable, some may not be, but the goal of this process, the limit-world to which a methodologically admissible inquiry will converge is not *causally* involved in the process itself. It is ontologically ideal.

The question of the causal origins of the transformations of belief in a methodologically appropriate inquiry is thus a useful way to focus the dispute over the ontology of inquiry which idealists and realists as we have described them are concerned with. But precisely this (hypothetical, since no detailed causal relation has been suggested) clarification of the vague talk of what world one "inhabits" enables us to see that the dispute between ontological idealists and realists, and hence the concern with the causal antecedents of successful inquiry, *do not* provide arguments for the standardist position. Ontological idealism, as was noted before, clearly favors giving full ontological status to non-standard worlds. For belief-worlds, which are non-standard worlds, are taken by them to be efficient causes of inquiry (i.e., are taken to be real ontologically). Also, the idealist has no trouble envisaging convergence to non-standard limit-worlds, as we saw to be the case with Peirce. But what reason does the ontological realist have to insist that the efficient cause of successful inquiry, according to this view, the limit-world, is standard? Quasi-objects, in the technical sense we have given to that term, can have *causal* properties just as they can have others, that is, according to the logic we have described. Quasi-objects were constructed precisely so that they could have whatever sorts of properties standard objects can have (although of course the quasi-objects and their properties and relations do not behave in logically standard ways). The realist's insistence that the causal motor of successful inquiry is the limit to which that inquiry will converge does not thereby determine anything about the standardness or non-standardness of that world, but establishes merely the conditional that *if* this world is non-standard, its quasi-objects must bear certain causal relations to the inquiry. Focussing our concern on specifically *causal* relations clarifies the dispute between the realists and the idealists. But it is ultimately irrelevant to the question of the standardness of whatever ends up being promoted as real, i.e., as the causal motor of successful inquiry.

Of course even if this were not the case—even if the invocation of causal relations did somehow determine that the real must be standard in its causal relations—this would not show that it was standard *tout court*. The real world might be standard in just those causal respects which drive correct inquiries, and still be non-standard with respect to other properties irrelevant to that process. But this is not a possibility which can be examined further in the absence of some particular argument about the necessity of causal standardness for reality. This is just the argument which the standardist is in need of—mere invocation of the causal order does not provide it. Our treatment of non-standardness applies equally well to causal properties and relations as to others.

Although we clearly cannot canvass all possible standardist strategies of argumentation, our constructions throughout have been aimed at undercutting one or another line of reasoning which might lead one to conclude that non-standard possible worlds are ontologically second class citizens. It may be that a standardist defense can yet be mounted on the basis of the representing/represented distinction, but it is clear that none of the most natural ways of doing so cut against the ontological equality of non-standard worlds and their denizens.

One last dialectical attack open to the standardist ought to be considered, however. Its point of departure lies in the very characteristics of our description of non-standard entities which we have employed against the standardist. For it might be objected that our whole scheme for presenting non-standard worlds itself subordinates, indeed *reduces* non-standard constructions to standard ones. The interpreted propositional calculus with which we began constructs non-standard, (i.e., weakly consistent) sets of sentences by the schematization and superposition of standard (maximal consistent) sets of sentences, and we proceeded similarly with their semantic interpretation. The extension to modal languages constructed the accessibility structure of the domain of non-standard possible worlds out of that of standard ones. The model theory which extended our results to quantified languages with identity operated in the same way. Standard model structures with domains of standard individuals formed a *basis* for the non-standard ones, and even when we moved to the lattice of non-standard possible worlds within which standard worlds were asserted to constitute merely a special degenerate case, we saw that it was possible to *resolve* each non-standard world into a product by schematization and superposition of standard worlds. Our whole procedure, so this objection runs, while seeking to show the ontological equality of non-standard worlds has in fact shown their subordination as clearly as it could be shown.

The description of our procedure embodied in this account is clearly correct, but the conclusion is a non sequitur. Let us look at the most detailed example, the model theory for weakly consistent theories of a quantified language with identify. Standardism has been presupposed by classical model theory. A model was taken to consist of a domain and a set of relations on that domain. A criterion of adequacy on the specification of a domain was that each element of that domain be *nameable* (though of course they do not all need to be named). This condition entails the standardness of the model theory in virtue of the identity conditions on objects which nameability entails, the standard laws of identity. In order to describe quasi-objects, which have ambiguous and overdetermined identity relations in a *standard* metalanguage, it was necessary to refer to the quasi-objects in a non-canonical way, by some means which would not entail standardness. The trick used was preliminary element-specifications and subsequent explicit identity relations. The procedure might be likened to that in which one tries to count black snakes (quasi-objects) in a dark room by means of luminous numerals (element specifications) which have been painted on them, not necessarily one per snake. One would need an explicit identity relation, saying which numerals appeared on the *same* snake in order to count them. You can't tell the players without a program. Of course quasi-objects must be thought of as non-standard snakes. Similar heroic referential maneuvers were required to capture the relations between quasi-objects. In this case the strategem involved the multiple partitioning of those relations.

These drastic measures were taken precisely so that the meta-language in which the model theory is couched could be a *standard* one, in which the syntactic objects of that meta-language would obey standard identity conditions, and the whole meta-theory would be consistent and complete. The virtue of our account in its details is that it *is* expressed in standard terms. Accordingly it does not beg important questions in the manner inevitable if one tried to recommend an *inconsistent* inconsistency tolerating logic over a consistent logic which does not tolerate inconsistencies. The way the standardness of the metalanguage was achieved was just by putting all of our assertions about non-standard worlds and their constituent quasi-objects and relations in *terms* of operations on standard worlds, objects, and relations. The model theory thus operated at one remove from the models it talks about. The structures we wrote down, with element-specifications, explicit identity relations, partitioned relations and all the rest, were not the models of weakly consistent theories. They *referred* to those models, they *specified* or *expressed* them. Strictly speaking one never writes down an algebraic structure, but only a name or description of one. But usually the relation between the designation of a model one does write down and the model structure referred to is much more straightforward than it is in the model theory we presented. It was this non-canonical interpretation of the metalanguage model-designations (which we can be perfectly precise about) which enabled a non-standard model theory to be couched in a syntactically standard meta-language.

The point is that the construction of non-standard structures out of standard ones is *not* a feature of those structures themselves, but only one of the means which we have chosen to represent those non-standard structures. It is a feature of the notation, not necessarily of the subject matter. Quasi-objects do not have the same structure as element-specifications, nor do their relations have the same structure as the predicates we constructed. We have shown how to use elemnet-specifications and partitioned predicates to talk usefully about quasi-objects and their relations, but not every feature of the former is a feature of the latter.

In particular, while our *designations* of non-standard worlds and their inhabitants are constructed by operations on their standard counterparts, no conclusion about the *ontological* constitution of the non-standard entities is to be drawn from that fact. Our theories about the physical world are composed of arrays of alphanumeric symbols, but we do not conclude from that that the world we describe with them is similarly constructed of alphanumeric symbols of their counterparts. We were able to use a syntactically standard metalanguage to couch a non-standard theory in virtue of two characteristics it was constructed to have. First, designations of non-standard elements are built out of designations of standard counterparts in a way which is describable in a standard theory. Second, the relation between our designations of non-standard objects and those objects themselves was not rigorously formalized (which would have required a non-standard theory) but was described precisely but in informal terms. The result is a description of non-standard entities which is built up out of descriptions of standard ones in definite and standardly describably ways. The non-standard entities themselves need exhibit none of this structure. So in spite of the reductive appearance of our exposition of the behavior of inconsistent and incomplete entities, their ontological status relative to their consistent and complete counterparts is not prejudiced thereby. Insofar as we have been successful in undermining standardist arguments by showing how inconsistencies may be logically housebroken, our employment of standard metalanguage should not be taken as rendering that victory empty.

SECTION 26

Conclusion: The Import of Inconsistency

The preceding discussion has explored a theory of schematic and superposed worlds in an endeavor to forge a possible-worlds semantics that can accommodate a Meinongian theory of objects and lay the groundwork for a more tolerant perspective on inconsistency. In doing this, it has sought to treat inconsistency as a localizable anomaly rather than an all-vitiating catastrophe. We have seen in detail how inconsistency may be accommodated in our deductions within a logical system and in the semantical interpretations attributed to such a system. The apparatus of non-standard possible worlds has been shown capable of interpreting inconsistent modal discourse, as well as the inconsistent and ambiguous relations of identity and individuation which characterize the Meinongian individuals which inhabit non-standard possible worlds. Finally, an ontological parity between standard and non-standard possible worlds was argued from the various ways in which ideal cognitive inquiries could be expressed and discussed in terms of convergence on the lattice of all possible worlds, be they standard or non-standard.

Perhaps the most important aspect of these deliberations relates to the orthodox view of inconsistency as an absolute and total epistemic disaster that immediately blocks all further prospect of rational dealings. This "standardist" perspective—as we have called it—would set consistency completely apart from all the other parameters of cognitive systematization. Unifying interlinkage, coherence, completeness, simplicity, explanatory adequacy, and the rest are all clearly matters of degree. Consistency alone—so it is held—differs from this general pattern in being an absolute requirement exempt from the give and take of a degree-admitting factor capable of entering into the process of rational cost-benefit analysis that can trade a quantity of this desideratum against a quantity of that. Inconsistency is not considered by the standardist to be a matter of degree; it is assigned an *infinite* negative weight in the rational assessment of cognitive success. No combination of epistemic virtues is allowed to mitigate the total systematic disaster which is signalled by the presence of inconsistency.

The present perspective indicates that both these aspects of the standardist perspective are open to serious doubt. In its suggestion that consistency too is a matter of degree, it puts all of the parameters of cognitive systematization on a par in this important regard. It too can thus enter into the cost-benefit calculation that must be carried out when rival possibilities arise in our endeavors at the achievement and systematization of knowledge. No single parameter of evaluation of such a systematization need be given infinite weight. Inconsistency has accordingly been removed from its status as an object of an *odium metaphysicum*, signalled by the assignment to it of an *infinite* cognitive disutility which makes inconsistency strictly incomparable to any of the other parameters we use to control our epistemic undertakings.

The toleration of inconsistencies within the sphere of rational systematization is not only *permissible*, but in suitable circumstances it may be *advantageous* and perhaps even unavoidable.

The availability of this resource has important ramifications for the theory of cognitive systematization. For systematicity is a crucial standard of acceptability, a key criterion of truth in the theory of knowledge.[102] And once one admits that consistency just is simply another mode of order or system, one that is not an absolute requisite but one desideratum among others and capable of being counterbalanced by these others (e.g., comprehensiveness), then our whole view of the nature of systems of knowledge is importantly transformed.

Nothing in these deliberations goes against regarding inconsistency as a negative factor—an emphatic liability or demerit. But it is not—or need not be—viewed as an absolute and decisive disqualification, one quite different in nature from such other cognitive non-desiderata as complexity or non-uniformity. Nothing that has been said here countervails against the standing of consistency as a desideratum of great weight and worth. But our considerations indicate it is just that—a *desideratum*. It need not be viewed (as has generally been the case) as a *necessitatum*, a requisite whose standing is absolute and in whose absence the book is simply closed on all prospects of rational discussion.

To be sure, we have emphatically *not* attempted to portray inconsistency as something whose acceptance is inevitable or intrinsically desirable. Rather, the aim has been to show that, even though inconsistency in our cognitive systematizations is not to be welcomed, its acceptance is at any rate *available* as a conceptual recourse, and one which may—in certain conceivable circumstances

—yield theoretical advantages sufficient to offset the negativities involved.

No doubt it is important in the interests of rationality to keep *our own* claims about a given subject consistent. But this is no reason to insist on consistency in the context of the *object* of our assertions. It is a key upshot of these deliberations that an inconsistent world can be discussed in perfectly consistent terms.

An important safety-valve in this regard is provided by distinction between (1) discourse at the level of an object language, or in the present context, ground-level descriptions of a world, and (2) discourse at the theoretical meta-level. Specifically we must distinguish between

(1) $t_s(\mathrm{p})\ \&\ t_s(\sim p)$

which, as we have seen, is a perfectly feasible circumstance in the case of an inconsistent system *S* (and does not—or need not— lead to $t_s(\mathrm{p}\ \&\ \sim\mathrm{p})$), and

(2) $t_s(p)\ \&\ \sim t_s(p)$.

For with (2)—a claim that itself fits the form $t(p\ \&\ \sim p)$—it is our own discourse that is inconsistent, whereas with (1) we have safely managed to insert another assertor—the system *S*—between ourselves and the inconsistency at issue. A claim of this second sort would indeed be problematic, for we would be inconsistent in *our own* assertive commitments of the ground-level system that it both does and does not have a certain feature. But a system can itself be inconsistent—and can be recognized by one as such, as per (1), without this inconsistency spilling over into our discourse about it as per (2). The inconsistency of the objects of our discussion—or indeed even of the world that we inhabit—need not affect the consistency of our own discourse. A consistent account of an inconsistent object of consideration is perfectly possible, and so is a perfectly consistent and coherent discussion of an inconsistent and incoherent subject-matter.

There is an important difference—one that must be duly recognized and preserved—between

(1) a consistent picture of a world: a world described consistently, a description of a world that is consistently given

and

(2) a picture of a consistent world: a description of a world that is consistent, a world described as being consistent.

There is, that is to say, a critical difference between a consistent world-description and a consistent-world description. (The placement of that hyphen matters a lot.) Even if the world is inconsistent, we want to be in a position to give a consistent picture of it. Self-consistency—the consistency of our own assertions—can remain an important methodological desideratum even for a Heracleitean who recognizes the inconsistency of nature. It is this stance in metaphysics and epistemology that underlies the logical policy of distinguishing between (1) and (2) above. In the former case we are describing an inconsistent world; in the latter we have ourselves crossed the borderline into inconsistency.

Now someone might well argue as follows:

> Why attach such importance to the consistency of a theory of the inconsistent? Once one is prepared to concede that there might be real contradictions, why not permit them to surface in our own deliberations? Why all this stress on the consistency of our own commitments in the context of a potentially inconsistent world?

The response here lies in *methodological* considerations. The aim of the present discussion is to show that *there simply is no need* for us to lapse into inconsistency ourselves in order to realize those objectives for which inconsistency-tolerant logics have traditionally been devised. We can have our cake and eat it too—i.e., we can handle a theory of inconsistent worlds in terms of a body of machinery that is itself inconsistency-shunning (indeed *classical*!). The traditional regulative maxim of rational inquiry ("Keep your own discussion consistent") is not something we need abandon: even if the world is inconsistent, we can reason about it in a self-consistent manner. An inconsistency in our own assertions simply betokens that "we do not know what we are talking about" here, and there is no good reason why this should ever be. We need not ourselves jabber confusedly in the face of a confusing situation nor inconsistently in the face of an inconsistent one. It is difficult to deny that consistency is a *desideratum*—though one may view it as something that may have to be abandoned in the face of recalcitrant circumstance. The present deliberations show that this unhappy result is not forced upon us—that we need never abandon consistency to achieve the ends at issue in our inquiries. It shows that the radically unorthodox ends of an inconsistency-tolerant approach can be achieved by very conservative means on the side of our logical commitments.

What is at issue here is an inversion of the usual perspective—the view that while man's (imperfect) thinking about the world may be inconsistent, the world itself must at any rate be consistent. We hold that ontologically speaking the world (or a world) may very possibly be *inconsistent* "in itself," but that nevertheless one should hew to the crucial methodological desideratum that *our thought* about things (even about inconsistent things) should be consistent. Consistency, in short, may figure less as a constitutive principle at the level of ontological world-description than as a regulative principle at the epistemological level of man-contrived inquiry. It is a prerequisite for the conduct of workable communication but not a descriptive requirement we can say *a priori* to be satisfied by the world about which we endeavor to communicate.

It is thus necessary to stress a point whose importance cannot be overemphasized. It would be a grave misreading of the situation to construe the present approach to the tolerance of inconsistency as succumbing, or even making concessions to unreason and irrationality.

Traditionally, inconsistency has been taken as the mark of the irrational, as its metaphysical essence. Theorists who were prepared to countenance inconsistency, claiming it as an ineluctable characteristic of thought and world, have typically coupled that attitude with the observation that boundaries are thereby drawn to the employment of reason—whether this conclusion is couched as the discovery that unreason in the form of inconsistency is to be overcome by turning our attention from the realm of mere irrational appearance to that of an intelligible Reality (as Bradley would have it) or as the claim that the contradictions thus intractable to reason must be transformed by revolutionary political praxis (as Engels prefers). Aid and comfort is given to inconsistency as tolerable and perhaps inevitable only insofar as that inconsistency is viewed as the ally of the irrational and intrinsically unintelligible.

But no attitude could be further from that of the present study. The central purpose of this discussion has been to establish the possibility of a rigorously rational treatment of inconsistency-admitting systems and the inconsistent objects they treat. Its prime concern has been to show that inconsistencies of various sorts are perfectly amenable to reasoned treatment.

The identification of inconsistency and irrationality is merely a more insidious version of standardist's exclusion of inconsistency from reality. The standardist takes inconsistency to be a cognitive killer-virus, whose appearance in a cognitive enterprise instantly infects its most distant elements, creating an epidemic of unreason which can end only in the destruction of the entire system in which it appears. We have sought to counter this superstitious attitude toward inconsistency (revealed by the untoward assignment of infinite negative weight to a single dimension of cognitive evaluation) by providing effective quarantine procedures for isolating inconsistency when it does appear, insuring that such an outbreak need not result in an all-consuming plague of irrationality. Where earlier logicians knew of no prudent response to the appearance of a single case of infection with inconsistency save to burn the entire village and move on, our own days have seen the development of methods of treatment which will prevent such an outbreak from spreading. We are now at last in a position to investigate possible beneficial side-effects of local infection with this particular disease organism.

The standardist sees consistency as the last bastion of cognitive *responsibility*. If this ultimate duty of reason were abrogated, none would remain. The result would be the epistemological anarchism of a nihilism whose barren slogan is "Everything is permitted." A recognition of the fallibilism that attends even our most successful empirical inquiries has freed us from dogmatic *a priori* constraints on the *content* of the cognitive systems we deploy in inquiry. The standardist is worried that the recommended demise of the remaining minimal *formal* constraint of consistency leaves cognitive inquiry entirely without discipline. But in cold fact, the distinction between reason and unreason can be decoupled from that between consistency and inconsistency. Our discussion shows that one can maintain as rigid a line as ever between rationality and irrationality even in the face of inconsistency. It must simply be recognized that, as regards (localizable) inconsistency, this line must be drawn in a somewhat different place. Inconsistency can be tolerated in the objects of thought and assertion, while, ultimately, discussion about them *can and should* be consistent at the meta-level of our cognitive commitments—in roughly the sort of way we have endeavored to indicate. The effect of our theory is thus not a disparagement of rationality and not a diminution but an enlargement of the boundaries of reason—an extension that encompasses territory previously deemed inaccessible.

APPENDIX I

Syntax of the Metalanguage of Non-Standard Possible Worlds

At several places in the text, recursions on the complexity of construction of a non-standard possible world are appealed to. This appendix offers a rigorous specification of the background of those recursions. As a starting point it is assumed that we have a set of expressions SPW informally interpreted as referring to standard possible worlds. We treat these as atomic expressions, and define in terms of them molecular expressions which refer to strictly non-standard possible worlds.

(A) NSPW is the smallest set of expressions (by inclusion) such that:

(1) If $w \in$ SPW, then $w \in$ NSPW

(2) If $a_1, a_2, \ldots a_n, \ldots$ is a finite or denumerably infinite sequence of expressions of NSPW, then $(\dot{\cup}_{i=1} a_i)$, an abbreviation for $(a_1 \cup a_2 \cup \ldots a_n \cup \ldots)$, is an expression of NSPW.

(3) If $a_1, a_2, \ldots a_n \ldots$ is a finite or denumerably infinite sequence of expressions of NSPW, then $(\dot{\cap}_{i=1} a_i)$, an abbreviation for $(a_1 \cap a_2 \cap \ldots a_n \cap \ldots)$, is an expression of NSPW.

(B) If a,b are two expressions of NSPW, we say that they are *co-constructional*, and write $a = b$ if and only if a can be transformed into b by a denumerable number of applications of transformations from the following list:

(i) For $w_1, w_2 \in$ SPW, if $w_1 = w_2$ according to the criteria of identity appropriate to SPW, replace w_1 by w_2 anywhere.

(ii) For $a_1, a_2 \in$ NSPW, replace $a_1 \cup a_2$ by $a_2 \cup a_1$ anywhere (note that here and in the subsequent clauses of this definition, the expressions which may be substituted for one another are *nearly well-formed*, in that they would be turned into well-formed expressions of NSPW by the addition of external parentheses).

(iii) For a_1, a_2 NSPW, replace $a_1 \cap a_2$ by $a_2 \cap a_1$ anywhere.

(iv) For $a_1,a_2, \ldots a_n, \ldots$ a finite or denumerably infinite set of expressions of NSPW, replace $\bigcup_{i=1}^{N} a_i$ by $a_1 \cup (\bigcup_{i=2}^{N} a_i)$ anywhere, or vice versa. N here can be any finite number or omega.

(v) For $a_1,a_2, \ldots a_n \ldots$ a finite or denumerably infinite set of expressions of NSPW, replace $\bigcap_{i=1}^{N} a_i$ by $a_1 \cap (\bigcap_{i=2}^{N} a_i)$ anywhere, or vice versa.

APPENDIX II

Algebraic Properties of Extended Accessibility Relations

This Appendix will show that if the accessibility relation R on standard worlds is reflexive, symmetric, or transitive, then the constructed accessibility relation R' on non-standard worlds is similarly reflexive, symmetric, or transitive.

Reflexivity. We proceed by recursion on the complexity of non-standard possible worlds. For the basis worlds (the standard, consistent and complete worlds from which non-standard worlds are constructed by superposition and schematization), if wRw, then $wR'w$ by clause (1) of the definition of R'. Suppose now that $w = w_i \frown w_j$. We want to show that if R' is reflexive at w_i,w_j, then it is reflexive at w. But it follows from clause (2) of the definition of R' that $wR'w$ holds if $w_iR'w_i$ and $w_jR'w_j$. Similarly for superposed worlds, by clause (3) of the definition of R'.

Symmetry. We proceed as before. For standard basis worlds w_1,w_2, by clause (1) of the definition of R': $(w_1R'w_2 \rightarrow w_2R'w_1)$ if $(w_1Rw_2 \rightarrow w_2Rw_1)$. Let $w_1 = w_i \frown w_j$ and $w_2 = w_k \frown w_m$. By clause (2) of the definition of R', then, it is clear that if $(w_iR'w_k \rightarrow w_kR'w_i)$ and $(w_jR'w_m \rightarrow w_mR'w_j)$, then $(w_1R'w_2 \rightarrow w_2R'w_1)$. The analogous result for superposed world follows from clause (3) of the definition of R'.

Transitivity. By clause (1) of the definition of R', transitivity of R' for base worlds is equivalent to transitivity of R. Let $w_1 = w_i \frown w_j$, $w_2 = w_k \frown w_m$, $w_3 = w_n \frown w_p$. Then by clause (2), if $(w_iR'w_k$ & $w_kR'w_n \rightarrow w_iR'w_n$ and $w_jR'w_m$ & $w_mR'w_p \rightarrow w_jR'w_p)$, then $(w_1R'w_2$ & $w_2R'w_3 \rightarrow w_1R'w_3)$. Thus transitivity holds at w_1,w_2,w_3 if it holds at the worlds they were constructed out of. The same argument, invoking clause (3) instead of clause (2) clearly applies to superposed worlds as well.

APPENDIX III

Completeness of the Non-Standard Modal Semantics

This Appendix will show that our implicit semantics for modal claims, formulated in terms of the superposition and schematization of possible worlds, is equivalent to the Kripke semantics for those modal claims, given the special accessibility relation defined for non-standard possible worlds by clauses (1)–(3) of the definition of R′ in section 14. Statements (i)–(iv) at the opening of that section codify necessary and sufficient conditions for modal propositions to be evaluated as true at a non-standard world according to our initial explanation of such worlds. We show that for each of these four cases, exactly the same valuations are generated by the Kripke definitions ((a) and (b) of section 14) given the special definition of the extended accessibility relation.

Case (*i*): By our definition of non-standard worlds, treating modal propositions just like any others, $/\Diamond p/_{w_1 \cup w_2} = T$ iff $/\Diamond p/_{w_1} = T$ or $/\Diamond p/_{w_2} = T$. By Kripke's account $/\Diamond p/_{w_1 \cup w_2} = T$ iff there is some world w such that $(w_1 \cup w_2)\mathrm{R}'w$ and $/p/_w = T$. We must show that these two specifications coincide, given our account of the extended accessibility relation R′. The proof is inductive. The equivalence holds trivially for standard basis worlds. We show that if that equivalence holds for worlds of a given constructional complexity, then it holds for all worlds which are superpositions or schematizations of worlds of that level of complexity (in case (i) we are looking only at superpositions). By clause (3) of the definition of R′, the Kripke conditions for $/\Diamond p/_{w_1 \cup w_2} = T$ will hold just in case there are w_k, w_m such that $w_1\mathrm{R}'w_k$ and $w_2\mathrm{R}'w_m$ and such that $/p/_{w_k \cup w_m} = T$. By our initial definition of $w_k \cup w_m$, $/p/_{w_k \cup w_m} = T$ iff $/p/_{w_k} = T$ or $/p/_{w_m} = T$. Suppose that $/p/_{w_k} = T$ (re-letter if necessary). Then since we stipulated above that $w_1 R' w_k$, by clause (b) of the Kripke definition $/\Diamond p/_{w_1} = T$. By our definition of truth in non-standard worlds, this is sufficient for $/\Diamond p/_{w_1 \cup w_2} = T$, which was to be shown.

For the converse (that if our conditions are met, so are Kripke's), we know by (i) that $/\Diamond p/_{w_1 \cup w_2} = T$ iff $/\Diamond p/_{w_1} = T$ or $/\Diamond p/_{w_2} = T$. Reletter so that $/\Diamond p/_{w_1} = T$. Then by clause (b) of the Kripke definition and our inductive hypothesis, there is some possible world

w_1 such that $w_1R'w_i$ and $/p/_{w_i} = T$. Let w_j be any world accessible from w_2. (We here assume that every world has some world accessible from it. This assumption, which is equivalent to the claim that $\Box p \rightarrow \Diamond p$ according to the Kripke interpretive schema (a) and (b) has been made, so far as I know, by every writer on intensional logic who has used the Kripke apparatus at all, so our reliance on this principle (which will be invoked again in case (iv) below) does not limit the generality of our result significantly. It is easy to show, using the methods of Appendix II, that if the original accessibility relation R on standard worlds meets this condition, then so will the constructed R′.) Then by our construction of the world $w_i \dot{\cup} w_j$, $/p/_{w_i \dot{\cup} w_j} = T$, since anything true in *either* of w_i or w_j is true in $w_i \dot{\cup} w_j$. But since by hypothesis $w_1R'w_i$ and $w_2R'w_j$, clause (3) of the definition of R′ ensures that $(w_1 \dot{\cup} w_2)R'(w_i \dot{\cup} w_j)$. Thus there is a world, namely $w_i \dot{\cup} w_j$, accessible from $w_1 \dot{\cup} w_2$, and in which p is true. Thus the Kripke conditions for the truth of $\Diamond p$ at $w_1 \dot{\cup} w_2$ hold if our non-standard conditions hold.

Case (*ii*): On Kripke's account, $/\Box p/_{w_1 \dot{\cup} w_2} = T$ iff for all worlds w such that $(w_1 \dot{\cup} w_2)R'w$, $/p/_w = T$. By clause (3) of the definition of R', this last condition is equivalent to the claim that for all w_k,w_m such that $w_1R'w_k$ and $w_2R'w_m$, $/p/_{w_k \dot{\cup} w_m} = T$. This condition will hold iff either for all w such that $w_1R'w$ $/p/_w = T$ or for all w such that $w_2R'w/p/_w = T$. For suppose that this were not the case, that is, that there were w_i,w_j such that $w_1R'w_i$ and $w_2R'w_j$ and $/p/_{w_i} \neq T$ and $/p/_{w_j} \neq T$. Then by construction according to our definition of superposed worlds, $/p/_{w_i \dot{\cup} w_j} \neq T$, contrary to our hypothesis. So either for all w such that $w_1R'w/p/_w = T$ or for all w such that $w_2R'w/p/_w = T$. Reletter if necessary so that for all w such that $w_1R'w/p/_w = T$. Then by Kripke's principle (a), $/\Box p/_{w_1} = T$, and so by our definition of superposition, $/\Box p/_{w_1 \dot{\cup} w_2} = T$.

For the converse, (ii) says that $/\Box p/_{w_1 \dot{\cup} w_2} = T$ iff $/\Box p/_{w_1} = T$ or $/\Box p/_{w_2} = T$. Reletter if necessary so that $/\Box p/_{w_1} = T$. Then by the Kripke principle (a), for all w such that $w_1R'w/p/_w = T$. Then for any world w_k such that $w_2R'w_k$, $/p/_{w \dot{\cup} w_k} = T$ by our definition of superposition, so by clause (3) of the definition of R', for any world w_m such that $(w_1 \dot{\cup} w_2)R^1w_m$ we have that $/p/_{w_m} = T$, which is just the Kripke condition for $/\Box p/_{w_1 \dot{\cup} w_2} = T$.

Case (*iii*): On Kripke's account, $/\Diamond p/_{w_1 \dot{\cap} w_2} = T$ iff there is a w such that $w_1 \dot{\cap} w_2R'w$, and $/p/_w = T$. By clause (2) of the definition of R', this will hold iff there are suitable w_i,w_j such that $w_1R'w_i$ and

$w_2R'w_j$ and $/p/_{w_i \cap w_j} = T$. By our definition of schematization, this will be the case iff $/p/_{w_i} = T$ and $/p/_{w_j} = T$. But if there are such w_i,w_j, then by Kripke's principle (b) $/\Diamond p/_{w_1} = T$ and $/\Diamond p/_{w_2} = T$, and this is just what our definition of schematization requires for $/\Diamond p/_{w_1 \cup w_2}$.

For the converse, by our definition of schematization $/\Diamond p/_{w_1 \cap w_2} = T$ iff $/\Diamond p/_{w_1} = T$ and $/\Diamond p/_{w_2} = T$. By Kripke's principle (b), this is equivalent to the existence of w_i,w_j such that $w_1R'w_i$ and $/p/_{w_i} = T$ and $w_2R'w_j$ and $/p/_{w_j} = T$. By our definition of schematization, then, $/p/_{w_i \cap w_j} = T$, and by clause (2) of the definition of $R'(w_1 \cap w_2)R'(w_i \cap w_j)$. Thus Kripke's condition for $/\Diamond p/_{w_1 \cap w_2} = T$ is satisfied. (It is obvious that we have been observing the requirements of constructional isomorphism all along in these proofs, and since we have been dealing separately with $w_1R'w_i$ and $w_2R'w_j$ the existence of suitable alphabetically canonical expressions to satisfy the definition of R' is assured, that is, we can simply substitute the alphabetically canonical expressions which are co-constructional with $w_1 \cap w_2$ and $w_i \cap w_j$ in this step.)

Case (iv): On Kripke's account, $/\Box p/_{w_1 \cap w_2} = T$ iff for all w such that $(w_1 \cap w_2)R'w_j\, /p/_w = T$. By clause (2) of the definition of R' this is equivalent to the condition that for all w_i,w_j such that $w_1R'w_i$ and $w_2R'w_j\, /p/_{w_i \cap w_j} = T$. By our definition of schematization, this will hold iff for all w_i such that $w_1R'w_i/p/_{w_i} = T$ and for all w_j such that $w_2R'w_j\, /p/_{w_j} = T$. By Kripke's principle (a), this will be true iff $/\Box p/_{w_1} = T$ and $/\Box p/_{w_2} = T$, which is just our definition of $/\Box p/_{w_1 \cap w_2} = T$.

For the converse, our definition of schematization will be satisfied iff $/\Box p/_{w_1} = T$ and $/\Box p/_{w_2} = T$, which by Kripke's principle (a) is just in case for all w_i,w_j such that $w_1R'w_i$ and $w_2R'w_j/p/_{w_i} = T$ and $/p/_{w_j} = T$, i.e. in case $/p/_{w_i \cap w_j} = T$. Then by clause (2) of the definition of R', this means that for all w such that $(w_1 \cap w_2)R'w$, $/p/_w = T$, which is just Kripke's condition for $/\Box p/_{w_1 \cap w_2} = T$.

APPENDIX IV

Satisfaction and the Superposition and Schematization Theorems

A. Satisfaction: We treat superposed structures first, and then indicate how that treatment must be altered for schematized ones. Let M^{12} be a superposed structure, let $\bar{z} = (x_1 \cup y_1, x_2 \cup y_2, \ldots x_n \cup y_n \ldots) = (z_1, z_2, \ldots z_n \ldots)$ be a countable sequence of element-specifications of the domain A^{12}, and let f, f_1, f_2 be well formed-formulae of the first-order language L. Then $M^{12} \models_{\bar{z}} f$ if and only if:

(i) $f = P_n(v_{k(1)}, v_{k(2)}, \ldots v_{k(d(n))})$ and $(z_{k(1)}, z_{k(2)}, \ldots z_{k(d(n))}) \in R_n^{12}$, where d is a function giving the adicity of the P_nth predicate for argument n, and k is a function assigning variables of L to natural numbers. Or

(ii) $f = (v_r = v_s)$ and $(z_r, z_s) \in I^{12}$. Or

(iii) $f = f \& f_2$ and $M^{12} \models^1_{\bar{z}} (f_1 \& f_2)$ or $M^{12} \models^2_{\bar{z}} (f_1 \& f_2)$.

This clause invokes a special notion which we have not yet defined, namely the notion of superscripted satisfaction $\models^1$ and $\models^2$. This notion will be defined below, and its significance will be discussed when we prove our theorems in this Appendix.

(iv) $f = (\forall v_n) f_1$ and for all $a \in A^{12}$, $M^{12} \models_{\bar{z}(n/a)} f_1$. Or

(v) $f = (\exists v_n) f_1$ and there is an $a \in A^{12}$, $M^{12} \models_{\bar{z}(n/a)} f_1$. Or

(vi) $f = \sim f_1$ and either (here we must consider six sub-cases):

- (a) $f_1 = P_n(v_{k(1)}, v_{k(2)}, \ldots v_{k(d(n))})$ and $(z_{k(1)}, \ldots z_{k(d(n))}) \in \bar{R}_n^{12}$ or
- (b) $f_1 = (v_r = v_s)$ and $(z_r, z_s) \in \bar{I}^{12}$ or
- (c) $f_1 = \sim f_2$ and $M^{12} \models_{\bar{z}} f_2$ or
- (d) $f_1 = f_2 \& f_3$ $M^{12} \models_{\bar{z}} \sim f_2$ or $M^{12} \models_{\bar{z}} \sim f_3$, or
- (e) $f_1 = (\forall v_n) f_2$ and there is some $a \in A^{12}$ such that $M^{12} \models_{\bar{z}(n/a)} \sim f_2$, or
- (f) $f_1 = (\exists v_n) f_2$ and there is some $a \in A^{12}$ such that $M^{12} \models_{\bar{z}(n/a)} f_2$.

The superscripted satisfaction function of clause (iii) is a way of mimicking the structure of M^1 and M^2 inside M^{12}. It is defined by the following six conditions (where j can be either 1 or 2 throughout, but always the same):

$M^{12} \vDash {}^{j}_{z} f$ if and only if

(1) $f = P_n(v_{k(1)}, v_{k(2)}, \ldots v_{k(d(n))})$ and $(z_{k(1)}, \ldots z_{k(d(n))}) \in R^{12}_n/_j$, or

(2) $f = (v_r = v_s)$ and $(z_r, z_s)I^{12}/j$, or

(3) $f = f_1 \& f_2$ and $M^{12} \vDash {}^{j}_{z} f_1$ and $M^{12} \vDash {}^{j}_{z} f_2$, or

(4) $f = \sim f_1$ and $M^{12} \vDash {}^{j}_{z} f_1$, or

(5) $f = (\forall v_n) f_1$ and for all $a \in A^{12}$, $M^{12} \vDash {}^{j}_{z(n/a)} f_1$, or

(6) $f = (\exists v_n) f_1$ and there is an $a \in A^{12}$, such that $M^{12} \vDash {}^{j}_{z(n/a)} f_1$.

The motivation of these definitions will be clear from the proof of the superposition theorem.

Satisfaction in schematized worlds is defined in a parallel fashion (substituting 'M_{12}' for 'M^{12}', 'R^n_{12}' for 'R^{12}_n', and so on throughout) except for clauses (iii) and (vid), which are replaced by:

(iii') $f = f_1 \& f_2$ and $M_{12} \vDash {}_{z} f_1$ and $M_{12} \vDash {}_{z} f_2$,

(vi–d') $f_1 = f_2 \& f_3$ and $M_{12} \nvDash {}^{1}_{z}(f_2 \& f_3)$ and $M_{12} \nvDash {}^{2}_{z}(f_2 \& f_3)$.

Thus superscripted satisfaction must be invoked for schematized structures in defining the *negations* of conjunctions, while in superposed structures it is required for the definition of the conjunctions themselves.

Our concern in the theorems below is to determine the relations between the sentences satisfied by M^{12} and M_{12} according to this definition of satisfaction, and the sentences satisfied by M_1 and M_2 according to the standard definition, to show that M^{12} deserves to be called the *superposition* of M_1 and M_2 and that M_{12} deserves to be called their *schematization*. Before doing so, however, it is worthwhile noticing that a slight loosening of terminology will allow us to evaluate isomorphs of standard worlds using this non-standard satisfaction relation. Thus, let us associate with each standard structure $M = (A, R_1 \ldots R_n)$ its *non-standard representative* M', defined as follows: A', the domain of M', consists of all elements specified by expressions of the form $a \frown a$, $a \in M$ (it doesn't matter whether we use '$\frown$' or '$\smile$' here), and a k-adic relation R'_n holds of a sequence of domain elements specified by $(a_1 \frown a_1, a_2 \frown a_2, \ldots a_n \frown a_n)$ just in case R_n holds of $(a_1 \ldots a_n)$ in M. Thus we can consider standard worlds as special, algebraically degenerate, cases of non-standard worlds. To exhibit this fact in such a way that our non-standard satisfaction relations apply directly, we must simply specify the standard worlds in the symbolism of M', rather than the more familiar form of M.

The *superposition theorem* (T1) maintains that for any formula f of the first-order language L, and for any standard structures M_1 and M_2 appropriate to L:

(T1) $M^{12} \models f$ iff $M_1 \models f$ or $M_2 \models f$

(where, as usual, $M \models f$ is an abbreviation of $(\forall \bar{z})M \models_{\bar{z}} f)$ or, put in terms of the text, $T(M^{12}) = T(M_1) \cup T(M_2)$. To show this will be to show that our algebraic construction of M^{12} in fact generates the semantic operation we have called 'superposition' throughout this work. A parallel result will then be demonstrated for schematization. The proof of T1 proceeds by induction on the length of f:

(i) Suppose $f = P_n(v_{k(1)}, \ldots v_{k(d(n))})$.

(a) Show that $(M_1 \models_{\bar{x}} f$ or $M_2 \models_{\bar{y}} f) \rightarrow M^{12} \models_{\bar{z}} f$, (where $\bar{x} = (x_1, x_2 \ldots x_n \ldots)$, each $x_i \in A_1$, $\bar{y} = (y_1, y_2, \ldots y_n \ldots)$ each $y_i \in A_2$, and $\bar{z} = (x_1 \cup y_1, x_2 \cup y_2, \ldots x_n \cup y_n \ldots)$, sometimes written $\bar{x} \cup \bar{y}$). We renumber if necessary so that $M_1 \models_{\bar{x}} f$. By the definition of satisfaction in standard structures, $(x_{k(1)}, \ldots x_{k(d(n))}) \in R^1_n$. Then by the definition of $R^{12}_n/1$ and R^{12}_n, for all $\bar{y} = (y_1, \ldots y_n \ldots)$, $(x_{k(1)} \cup y_{k(1)}, \ldots x_{k(d(n))} \cup y_{k(d(n))}) \in R^{12}_n/1$, and hence is also an element of R^{12}_n.

(b) Show that $(M^{12} \models_{\bar{z}} f) \rightarrow (M_1 \models_{\bar{x}} f$ or $M_2 \models_{\bar{y}} f)$. Define $\bar{z}$ as before. By the definition of satisfaction for M^{12}, $M^{12} \models_{\bar{z}} f$ iff $(x_{k(1)} \cup y_{k(1)}, \ldots x_{k(d(n))} \cup y_{k(d(n))}) \in R^{12}_n$. By the definitions of R^{12}_n, $R^{12}_n/1$, and $R^{12}_n/2$, this will hold iff *either* $(x_{k(1)}, x_{k(2)}, \ldots x_{k(d(n))}) \in R^1_n$ *or* $(y_{k(1)}, \ldots y_{k(d(n))}) \in R^2_n$. By the definition of standard satisfaction, this is just if $M_1 \models_{\bar{x}} f$ or $M_2 \models_{y} f$.

(ii) Suppose $f = (v_r = v_s)$. The theorem holds in this case as a special case of (i) above, substituting 'I^{12}' for 'R^{12}_n' throughout.

(iii) Suppose $f = f_1 \& f_2$. For this case we need to show as a lemma that superscripted satisfaction does in fact reconstruct M_j in M^{12}, i.e. that $M^{12} \models^j_{\bar{z}} f$ iff $M_j \models_{\bar{z}/j} f$ (where $\bar{z}/1 = \bar{x}$ and $\bar{z}/2 = \bar{y}$.) *Proof of lemma*: By clause (1) of the definition of superscripted satisfaction, and the definition of R^{12}_n/j, it is clear that for $f = P_n(v_{k(1)}, \ldots v_{k(d(n))})$, $M^{12} \models^j_{\bar{z}} f$ iff $(z_{k(1)}, \ldots z_{k(d(n))})/j \in R^j_n$, i.e. $M_j \models_{\bar{z}/j} f$. Again, identity can be treated as a special relation and subsumed under this case. But now clauses (3)–(6) governing superscripted satisfaction for non-atomic formulae are just the standard Tarski clauses. Since we have shown that j-superscripted satisfaction in M^{12} coincides with Tarski satisfaction in M_j for atomic formulae, the same must then hold true for *all* formulae.

Returning to clause (iii) of the proof of (T1), we use the lemma:

(a) Show that $(M_1 \models_{\bar{x}}(f_1 \,\&\, f_2)$ or $M_2 \models_{y}(f_1 \,\&\, f_2)) \rightarrow (M^{12} \models_{\bar{z}}(f_1 \,\&\, f_2))$: Renumber so that $M_1 \models_{\bar{x}}(f_1 \,\&\, f_2)$. By the lemma, this means that $M^{12} \models^{1}_{\bar{z}}(f_1 \,\&\, f_2)$, which by clause (iii) of the definition of non-standard satisfaction is sufficient for $M^{12} \models_{\bar{z}}(f_1 \,\&\, f_2)$.

(b) To show the other direction, by clause (iii) of our definition, there is a j such that $(\forall \bar{z})(M^{12} \models^{j}_{\bar{z}}(f_1 \,\&\, f_2))$. Renumber so that $j = 1$. Then by the lemma $M_1 \models_{\bar{x}}(f_1 \,\&\, f_2)$.

(iv) Suppose that $f = (\forall v_n)f_1$:

(a) Show that $(M_1 \models_{\bar{x}}(\forall v_n)f_1$ or $M_2 \models_{\bar{x}}(\forall v_n)f_1) \rightarrow (M^{12} \models_{\bar{z}}(\forall v_n)f_1$: Renumber so that $M_1 \models_{\bar{x}}(\forall v_n)f_1$. Then for all $a \in A_1$, $M_1 \models_{\bar{x}(n/a)}f_1$. By inductive hypothesis, then, for all $a \in A^{12}$ $M^{12} \models_{\bar{z}(n/a)}f_1$. Then by clause (iv) of the definition of non-standard satisfaction, $M^{12} \models_{\bar{z}}(\forall v_n)f_1$.

(b) For the other direction, suppose $M_1 \models_{\bar{x}}(\forall v_n)f_1$ and $M_2 \models_{\bar{y}}(\forall v_n)f_1$, and show that $M^{12} \models_{\bar{z}}(\forall v_n)f_1$: According to our inductive hypothesis and the definition of satisfaction for standard structures, $\exists a \in A_1\ \exists a' \in A_2$ such that $M_1 \models_{\bar{x}(n/a)}f_1$ and $M_2 \models_{\bar{y}(n/a')}f_1$. Let $a'' = a \cup a'$, then by inductive hypothesis $M^{12} \models_{\bar{z}}(n/a'')f_1$, which by clause (iv) of our definition shows that $M^{12} \models_{\bar{z}}(\forall v_n)f_1$.

(v) Suppose that $f = (\exists v_n)f_1$:

(a) Show: $(M_1 \models_{\bar{x}}(\exists v_n)f_1$ or $M_2 \models_{\bar{y}}(\exists v_n)f_1) \rightarrow (M^{12} \models_{\bar{z}}(\exists v_n)f_1)$. Renumber so that M_1 satisfies the formula. Then there is an $a \in A_1$ such that $M_1 \models_{\bar{x}(n/a)}f_1$. Define $\bar{z}$ as always, using arbitrary $\bar{y} \in (A_2)\omega$, and let $a'' = a \cup a'$ for some $a' \in A_2$. Then by inductive hypothesis, $M^{12} \models_{\bar{z}(n/a'')}f_1$, which by clause (v) of our definition is sufficient for $M^{12} \models_{\bar{z}}(\exists v_n) f_1$.

(b) For the other direction, suppose $M_1 \not\models_{\bar{x}}(\exists v_n)f_1$ and $M_2 \not\models_{\bar{y}}(\exists v_n)f_1$. Then there are no $a \in A_1$, $a, \in A_2$ such that $M_1 \models_{\bar{x}(n/a)}f_1$ and $M_2 \models_{\bar{y}(n/a')}f_1$. By inductive hypothesis, then, there is no $a'' \in A^{12}$ such that $M^{12} \models_{\bar{z}(n/a'')}f_1$, so $M^{12} \not\models_{\bar{z}}(\exists v_n)f_1$.

(vi) Suppose $f = {\sim}f_1$. Here we must consider six cases, corresponding to the six sub-clauses in part (vi) of our definition of satisfaction in non-standard structures.

(a & b) If $f_1 = P_n(v_{k(1)}, \ldots v_{k(d(n))})$ or $f_1 = (v_r = v_s)$, the proofs go through in both directions just as in (i) and (ii) above, substituting $\not{R}$ for R and $\not{I}$ for I throughout, since these converse relations are defined to hold just in case R and I do *not* hold in the standard worlds, which according to the definition of satisfaction in standard worlds is just in case $\sim f$ holds.

(c) (T1) follows if $f_1 = \sim f_2$ by our inductive hypothesis, together with the behavior of negation in standard structures.

(d) Suppose $f_1 = f_2 \,\&\, f_3$. We show that $\sim(f_2 \,\&\, f_3)$ holds in M^{12} just in case it holds in either M_1 or M_2. Suppose first that $M_1 \vDash_{\bar{x}} \sim(f_2 \,\&\, f_3)$. Then by the definition of standard satisfaction, either $M_1 \vDash_{\bar{x}} \sim f_2$ or $M_1 \vDash_{\bar{x}} \sim f_3$. By inductive hypothesis, then, either $M^{12} \vDash_{\bar{z}} \sim f_2$ or $M^{12} \vDash_{\bar{z}} \sim f_3$, and so by clause (vi–d) of our definition, $M^{12} \vDash_{\bar{z}} \sim(f_2 \,\&\, f_3)$.

For the other direction, if $M^{12} \vDash_{\bar{z}} \sim(f_2 \,\&\, f_3)$, then by (vi–d) either $M^{12} \vDash_{\bar{z}} \sim f_2$ or $M^{12} \vDash_{\bar{z}} \sim f_3$. Suppose the first holds. Then by inductive hypothesis, either $M_1 \vDash_{\bar{x}} \sim f_2$ or $M_2 \vDash_{\bar{y}} \sim f_3$. Whichever of these holds is sufficient for some $M_j \vDash_{z/j} \sim(f_2 \,\&\, f_3)$.

(e & f) These proofs are parallel to those of clauses (iv) and (v) above.

This completes the proof of (T1), establishing that a sentence is satisfied in M^{12} just in case it is satisfied by either M_1 or M_2. Since this was our original definition of superposition (in semantic terms), we are now entitled to call the algebraic operation defined in the text "superposition" as well.

Now we state and prove the parallel result for non-standard structures which are the result of the algebraic *schematization* of two standard worlds. We show that the sentences non-standardly satisfied by such a structure are just those sentences which are standardly satisfied by *both* the standard basis structures.

(T2) $M_{12} \vDash f$ iff $M_1 \vDash f$ and $M_2 \vDash f$

This result corresponds to the equation

$$T(M_{12}) = T(M_1) \cap T(M_2).$$

Proof of (T2) proceeds by induction on the length of f:

(i) Suppose $f = P_n(v_{k(1)}, \ldots v_{k(d(n))})$. By the definition of R^n_{12}, $(x_{k(1)} \frown y_{k(1)}, x_{k(2)} \frown y_{k(2)}, \ldots x_{k(d(n))} \frown y_{k(d(n))}) \in R^n_{12}$ if and only if $(x_{k(1)}, x_{k(2)}, \ldots x_{k(d(n))}) \in R^1_n$ *and* $(y_{k(1)}, y_{k(2)}, \ldots y_{k(d(n))}) \in R^2_n$, i.e., by the definition of satisfaction for standard structures, iff $M_1 \vDash_{\bar{x}} f$ and $M_2 \vDash_{\bar{y}} f$.

(ii) Suppose $f = (v_r = v_s)$, the argument of (i) above applies, with 'I_{12}' substituted for 'R^n_{12}' etc.

(iii) Suppose $f = f_1 \& f_2$. By clause (iii') of the definition of non-standard satisfaction for schematized structures, $M_{12} \models_{\bar{z}} f_1 \& f_2$ iff ($M_{12} \models_{\bar{z}} f_1$ and $M_{12} \models_{\bar{z}} f_2$), which by inductive hypothesis holds iff ($M_1 \models_{\bar{x}} f_1$ and $M_1 \models_{\bar{x}} f_2$ and $M_2 \models_{\bar{y}} f_1$ and $M_2 \models_{\bar{y}} f_2$). Since M_1 and M_2 are standard structures, this last is equivalent to $M_1 \models_{\bar{x}} f_1 \& f_2$ and $M_2 \models_{\bar{y}} f_1 \& f_2$.

(iv) Suppose $f = (\forall v_n) f_1$. If If $M_1 \models_{\bar{x}} (\forall v_n) f_1$ and $M_2 \models_{\bar{y}} (\forall v_n) f_1$, then by the definition of satisfaction for standard structures, $M_1 \models_{\bar{x}(n/a)} f_1$ for all $a \in A_1$, and $M_2 \models_{\bar{y}(n/a')} f_1$ for all $a' \in A_2$. Then by inductive hypothesis, setting $a'' = a \frown a'$, $M_{12} \models_{\bar{z}(n/a'')} f_1$, which by our definition of satisfaction for non-standard structures is equivalent to $M_{12} \models_{\bar{z}} (\forall v_n) f_1$. These steps are reversible, so both directions have been shown.

(v) Suppose $f = (\exists v_n) f_1$. If $M_1 \models_{\bar{x}} (\exists v_n) f_1$ and $M_2 \models_{\bar{y}} (\exists v_n) f_1$, then by the definition of satisfaction for standard structures, there is some $a \in A_1$ s.t. $M_1 \models_{\bar{x}(n/a)} f_1$ *and* there is an $a' \in A_2$ s.t. $M_2 \models_{\bar{y}(n/a')} f_1$. By inductive hypothesis, this is equivalent to there being an $a'' \in A_{12}$ s.t. $a'' = a \frown a'$ and $M_{12} \models_{\bar{z}(n/a'')} f_1$, which by the non-standard satisfaction definition is equivalent to $M_{12} \models_{\bar{z}} (\exists v_n) f_1$.

(vi) Suppose $f = \sim f_1$. There are six sub-cases, but all of them are straightforward adaptations of the corresponding proofs above except for (d), where $f_1 = f_2 \& f_3$, and we must show that $M_{12} \models_{\bar{z}} \sim (f_2 \& f_3)$ iff ($M_1 \models_{\bar{x}} \sim (f_2 \& f_3)$ and $M_2 \not\models_{\bar{y}} \sim (f_2 \& f_3)$). Assuming the left-most of these, clause (vi–d') of the satisfaction definition for non-standard structures yields ($M_{12} \not\models_{z}{}^1 (f_2 \& f_3)$ and $M_{12} \not\models^2_{\bar{z}} (f_2 \& f_3)$). In virtue of the lemma proved above as part of the proof of (T1) part (iii), this last conjunctive condition is equivalent to ($M_1 \not\models_{\bar{x}} (f_2 \& f_3)$ and $M_2 \not\models_{\bar{y}} (f_2 \& f_3)$), which by the standard satisfaction conditions for negation holds iff ($M_1 \models_{\bar{x}} \sim (f_2 \& f_3)$ and $M_2 \models_{\bar{y}} \sim (f_2 \& f_3)$).

This completes the proof of (T2), showing that, given our non-standard satisfaction relation, algebraically schematized structures yield semantic results identical to our original sense of "schematization."

* * *

In presenting our detailed account of the model-theory of superposition and schematization we made various simplifying assumptions. In this section we will indicate how some of the more important of these may be removed.

The first and most obvious restriction which we acknowledged during our demonstrations above was that we took superposition and schematization to be operations which applied to *pairs* of standard worlds to yield non-standard worlds. Nothing in the structure of our construction of non-standard worlds, our definition of the satisfaction relation appropriate to them, or the demonstrations of the central theorems (T1) and (T2) depended upon the limitation to *two* base worlds, however. In fact, our procedures apply equally well to the fusion by superposition or schematization of any countable number of standard worlds. The domain of a superposed world might consist of element-specifications of the form $x_1 \uplus x_2 \uplus x_3 \uplus x_4 \ldots$, where each x_i is an element of the standard world W_i, and the superposition which results is the world $W_1 \uplus W_2 \uplus W_3 \ldots$. The superposition theorem, which states that the set of sentences satisfied by such a world will be just the union of the sets of sentences satisfied by the countable number of base worlds, goes through with only the most minor of alterations.

Another issue which might arise here concerns the addition of *names* to the object language *L*. Formally, these present little problem. The satisfaction or interpretation function will assign them quasi-objects, referred to by means of the apparatus of element-specifications and explicit identity and non-identity relations, and then treat them just like variables under an assignment of values. In case a named object exists in two different base worlds, the name will have a bearer in both their superposition and their schematization. In each base we can take that bearer to be $a \uplus a$ ($a \cap a$ in schematized worlds), but must remember that this is a *quasi*-object, with ambiguous identity relations to $a \uplus b$ (or $a \cap b$). So the question arises as to the right sort of transworld identity relations to assert between standard objects, such as that referred to by '*a*' in our example, and quasi-objects composed out of them, such as '$a \uplus a$' in our example. It is clear that the ambiguous identity relations of quasi-objects rules out the application to them of a consistent and complete identity theory in a standard metalanguage, such as that to which accounts of trans-world identity typically aspire. Happily, standard objects can be treated as special cases of quasi-objects. They are just quasi-objects which obey stricter conditions of identity. Much will be made in the rest of the book about the fact that standard consistent and complete worlds are special cases of the more generally defined non-standard worlds, and that Tarskian satisfaction is a special case of non-standard satisfaction as we have defined it. A theory of the trans-world identity and individuation of quasi-objects (including standard consistent and complete objects, and their relations to other quasi-objects) must accordingly be interpreted on a master-domain which is the union of all non-standard domains. The theory of trans-world identity in such a domain will then of necessity use the apparatus of element-specifications and explicit identity and non-identity relations, just as were required within non-standard worlds. We will not present such a theory here, since no new principles, over and above those we have already examined for intra-world identity, are required for the apparently more ambitious undertaking, and so no greater understanding of quasi-objects emerges.

Another dimension along which we can generalize our previous discussion is the conditions we require our base worlds to meet. Up until this point, we have considered superposing and schematizing only sets of *standard*, consistent and complete, worlds. The *semantic* notions of superposition and schematization, corresponding to the union and intersection respectively of the sentences of various theories, however. There is no reason we can't consider the sentences satisfied by *both* of two superposed (and hence inconsistent) worlds. So we would like our algebraic notions of superposition and schematization to be similarly unrestricted in application. There is no fact no difficulty in principle in such a relaxation of our initial conditions, though the details are too cumbersome to be worth presenting in much detail. We appealed to the standardness of the base structures in two different places in the proof of (T1) and (T2). First, it is used in the treatment of superscripted satisfaction in the conjunction clause of the definition of non-standard satisfaction. Then standardness is invoked in the sub-clause governing negations of conjunctions. Looking at the function the assumption of standardness played in those two places will show us how that assumption can be dispensed with. The difficulty with conjunction is that in superposed worlds $f_1 \,\&\, f_2$ is satisfied if and only if that conjunction is satisfied in some one of the base worlds. There is in principle no way of moving directly from the satisfaction properties of f_1 and f_2 as separate sentences in a superposed world to the satisfaction properties of their conjunction in that world. To express this property algebraically in a way that could be recovered appropriately in a satisfaction relation which looks only at the superposed world and not at the base worlds which compose it, we in effect *built in* to the structure of non-standard worlds the structure of the base worlds they are constructed out of. That is, we had to be able to recover from the non-standard world alone information concerning what sentences had been satisfied by its base worlds. To do this, we stipulated a partition of the relations of our non-standard worlds in such a way that, together with out general account of the domains of non-standard worlds, the relations which held in the base structures are recoverable from the more complex relations of the non-standard worlds. Given this information and the standardness of the base worlds, we were able to use the Tarski definition of satisfaction to generate, as a sub-recursion within our definition of non-standard satisfaction, the sentences satisfied (standardly) in the base worlds, and hence could define conjunction appropriately. In order to free ourselves from the burdensome requirement of standardness of base worlds, we need only change the last step of this procedure. Instead of using the standard, Tarskian definition of satisfaction to determine what was true in the base structures (and hence what conjunctions should be satisfied in the non-standard world), we must simply use our own non-standard satisfaction definitions. Thus the definition of superscripted satisfaction will have to be adjusted so that the primary lemma concerning it still holds in the case in which M_j is a non-standard structure, and similarly for the account of the negation of a conjunction. The definitions and proofs we have presented are thus strictly appropriate only at the first level of construction of non-standard structures. They tell us about non-standard structures resulting from the superposition and schematization of sets of *standard* worlds. If we now take those definitions and substitute into the two treatments of conjunction mentioned above this first level account of non-standard satisfaction where Tarskian, standard satisfaction is now appealed to, the result will be an account of non-standard structures constructed by the super position and schematization of non-standard structures of the first level. Clearly our central theorems (T1) and (T2) will be demonstrable for this second level as well. Equally clearly, this process of extending non-standard structures to include the products of schematization and superposition of non-standard worlds of a lower level is recursively specifiable, since at each level one simply uses the disjunction of the non-standard satisfaction relations defined for previous levels in the two places where we originally appealed to Tarskian definitions. In the rest of this book, we will use "non-standard world" to refer to any structure of any level constructible ultimately from standard worlds by this process, and by "non-standard satisfaction" we will mean the disjunction of the definitions of satisfaction which emerge at each stage in the heirarchy of levels of non-standardness.

A final generalization of our basic construction may be mentioned, though it will not be exploited. There is no reason why we could not define non-standard worlds that were *mixed* in the sense of containing *both* elements of the form $x \cup y$ *and* elements of the form $x \cap y$. Our concern throughout has been with the superposition and schematization of *worlds*, and no such process as we have defined it will result in mixed worlds. But our model-theoretic account of these two modes of world-fusion ended by enabling us to speak of the fusion in this sense of the *individuals* inhabiting those worlds, and of the resulting quasi-objects. Our definition of satisfaction in non-standard worlds (the first-level account which we examined in detail, and which is the basis for the more general notion we have waved our hands at) applies straightforwardly to mixed worlds, once we have defined the relations which are to hold between the mixed element-specifications. We have no project to recommend which would motivate semantically the choice of one or another of the various ways of defining these relations, and so will leave the suggestion undeveloped.

APPENDIX V

Convergent Nets and Paths on a Lattice

The definitions of this appendix are all standard mathematical fare. See for instance G. Birkhoff's *Lattice Theory* (Volume XXV American Mathematical Society Series 1967, pp. 244ff.)

1. A lattice is *complete* iff in addition to satisfying the lattice axioms (L1)–(L4) stated in the text, every subset of the lattice has a meet and a join. In our case, this means that superposition and schematization must apply to arbitrary sets of worlds, not just countable sets. Our motivations and detailed implementations allow this, so we will take advantage of it.

2. An *upper-directed* set is a set I ordered by $\neq$ such that for any $i, j \in I$, there is a k such that $i \leqslant k$ and $j \leqslant k$.

3. A *net* is a set indexed by an upper-directed set. If the upper directed set is totally ordered, we may call it a *path*. In our applications, paths will be employed rather than the more general nets, and the ordering principle of their upper-directed index sets is taken to be *temporal*, so that progress along a path in a lattice corresponds to ever *later* stages in some inquiry. Our present interest lies in exploiting the formal relations between such an ordering principle and the order codified in the lattice structure induced by superposition and schematization.

4. That lattice-order, symbolized by $\leqslant$, is defined by $w_1 \leqslant w_2$ if and only if either $w_1 \cup w_2 = w_2$ or $w_1 \cap w_2 = w_1$. This induces a partial ordering on the lattice of non-standard possible worlds.

5. For any subset X of a partially ordered set, an *upper bound* of X is an element a such that for all $x \in X$, $x \leqslant a$. A *least* upper bound is an upper bound a of X such that for any other upper bound b of X, $b \leqslant a$. The greatest lower bound of X is defined dually. We write the least upper bound of X as *sup* X, and its greatest lower bound as *inf* X.

6. Given a net $\{x_i / i \in I$ an upper-indexed set$\}$, we define $\mathrm{Lim}(\inf\{x_i\}) =_{\mathrm{df}}. \mathrm{Sup} \leqslant \{y_j / y_j = \inf \leqslant \{x_i / i \geqslant j\}\}$, where sup $\leqslant$ indicates that it is the lattice ordering, not the index ordering, with respect to which we consider bounds. Similarly, we define $\mathrm{Lim}(\sup\{x_i\}) =_{\mathrm{df}}. \mathrm{Inf}. \leqslant \{y_j / y_j = \sup. \leqslant \{x_i / i \geqslant j\}\}$.

7. With these definitions in hand, we can say that a net $\{x_i/i \in I\}$ *converges to a* on a complete lattice (sometimes this is called *order convergence*) iff:

$$\mathrm{Lim}(\inf\{x_i\}) = \mathrm{Lim}(\sup\{x_i\}) = a$$

This is motivated by the familiar fact that for the real numbers the convergence of a sequence $\{x_i\}$ to the limit a is equivalent to the condition that $\lim(\sup\{x_i\}) = \lim(\inf\{x_i\}) = a$. Our definitions simply generalize convergence as we understand it on real numbers, with the index ordering being induced by the ordinality of various positions in the sequence of real numbers, and the analogue of our lattice ordering $\leqslant$ coming from the intrinsic ordering of the reals.

We may utilize these formal definitions so as to be able to define inquiry as a path from non-standard belief-world to non-standard belief-world, totally ordered by the relation "arrived at *after*", and thus be able to discuss the *convergence* of such indexed inquiries on the lattice of non-standard possible worlds with *its* ordering principle induced by the behavior of schematization and superposition.

NOTES

Section 1

1. A recent examination of Aristotle's case against the maintenance of inconsistent claims is R.M. Dancy, *Sense and Contradiction: A Study in Aristotle* (Dordrecht and Boston, 1975). This author is not impressed with the conclusiveness of Aristotle's justificatory discussions. "One might deny the law of noncontradiction for all sorts of reasons. None that I have seen strike me as good reasons. But neither do I see any reason for saying that there never *could* be a good reason for denying it." (P. 142.) Such a view is not seriously at odds with what is, in the final analysis, Aristotle's own position—viz., that the law is *axiomatic* in status (a prospect which Dancy is disinclined to take seriously). For if it is indeed the case that a thesis is properly to be classed as an axiom within a region of discourse, one would not expect to find these convincing arguments leading to it. Its justification would have to be forthcoming in terms of the systematic advantages of the over-all frame of thought to which it gives rise, and would emerge at the end rather than the outset of inquiry. And at this stage the supportive argumentation is not foundational, but at best the adducing of considerations of motivational plausibility designed to induce the reader's acceptance. (Axioms are starting-points in appearance only.)

2. From the *epistemic* point of view, it is significant that an abridgement of the Law of Contradiction lies deep in the nature of rational controversy as a mode of argumentation that is less than *totally* conclusive. We know that in the case of *deductively valid* arguments one cannot reason from true premisses to mutually inconsistent conclusions by the principle of classical deductive logic. But this is not so in argumentation of sub-deductive strength. Here it becomes entirely possible—in theory, at any rate—to build up highly cogent arguments for mutually inconsistent conclusions. When the premisses at our disposal are merely plausible or probable (rather than categorically true) and the modes of inference we use are ampliative and inductively strong (rather than logically airtight), it becomes altogether feasible to build up highly convincing arguments on the one side for P and on the other for $\sim P$. There are grounds (and not insubstantial ones at that) for thinking that any version of inductive inference construed along traditional, acceptance-oriented lines will give rise to inconsistencies. See Carl G. Hempel, "Inductive Inconsistencies," *Synthese*, vol. 12 (1960), pp. 439–469.

3. No doubt, *behavioral* "inconsistency" in cases affecting others—i.e., the refusal to "treat like cases alike" is reprehensible in an agent as violating fairness, but it is rather a mode of *disuniformity* than one of *self-contradiction.* We must not allow our moral distaste against *this* sort of inconsistency stand in the way of a more neutral rational attitude towards the other, logical and epistemological cousins.

Section 2

4. Already the earliest papers by Saul Kripke and John Lemmon on the relational semantics for modal logic allowed (strongly) inconsistent worlds in the treatment of the so-called "non-normal" modal logics (S1 and S2). However, the "worlds" of these discussions were a purely technical device that lacked any dimension of ontological relevancy.

Section 3

5. On other issues related to schematic worlds see N. Rescher, *A Theory of Possibility* (Oxford, 1975).

6. Compare also the view that statements about future contingencies are neither true nor false, but are of an indeterminate truth-status in lacking a truth value.

The world is accordingly schematic with regard to future contingencies. See the discussion of this theory in N. Rescher, "Truth and Necessity in Temporal Perspective" in *Essays in Philosophical Analysis* (Pittsburgh, 1969), pp. 271–302.

7. Or again, think of an equivocal message which—in the manner of some oracular deliverances—can be interpreted with equal validity in two mutually conflicting ways, both of which incompatible assertions are "intended"—the one for public (exoteric), the other for private (esoteric) consumption.

8. The teaching of the American New Realists took such a stance along these lines with respect to the sensory qualities of things in the face of discordant sense-perceptions:

> Oversimplifying slightly, we may say that Democritus reasoned in this way: "The wine that tastes sweet to me tastes sour to you; therefore, I do not perceive that it is sweet and you do not perceive that it is sour, and the wine itself is neither sweet nor sour." Protagoras, however, reasoned in a somewhat different way: "The wine that tastes sweet to me tastes sour to you; hence, I perceive that it is sweet and you perceive that it is sour; and therefore, one cannot say absolutely either that the wine is sweet or that the wine is sour; one can only say relativistically that whereas it is true for me that the wine is sweet, it is true for you that the wine is sour." And some of the American New Realists, in defense of the view that "things *are* just what they *seem*," drew still another conclusion: "The wine that tastes sweet to me tastes sour to you; therefore, one must say (absolutely and not relativistically) that there are contradictions in nature; one must say of the wine not only that it is both sweet and not sweet, but also that it is both sour and not sour." (Roderick M. Chisholm, *Theory of Knowledge* [Englewood Cliffs, 1966], p. 92.)

The idea that the wine has two incompatible tastes was, of course, *not* construed to mean that it had *every* taste: it is both sweet and sour, but not (say) fishy.

Section 4

9. These aspects of the schematic are developed more fully in section 20 below.

10. In the case of n worlds, a 2^n-valued logic can be used. The present discussion of the case $n = 2$ generalizes routinely.

11. This process of product-formulation for many-valued systems originated in a 1936 paper by Stanislaw Jaskowski (republished in English translation in S. McCall [ed]., *Polish Logic: 1920–1939* [Oxford, 1967]). Such systems are discussed in Nicholas Rescher, *Many-valued Logic* (New York, 1969), pp. 96–102.

12. Moreover, $+-$ is a different value from $-+$. World-fusion is not a wholly symmetric process: the order of composition matters. The contribution made by the components of a fusion world can only be determined synoptically and systematically—by having the ontological status of *all* propositions in the fused result.

13. But would it not be irrational ever to accept both P and $\sim P$ for the reason that one can always make a "dutch book" against a person who takes this stance?

This decision-theoretic counterargument is predicated on the idea that someone who accepts P will give odds on P in a bet on P vs. $\sim P$. And so, analogously, if one accepts $\sim P$ one will give odds on $\sim P$ in a bet on $\sim P$ vs P. Then, if we put $1 on P at (say) odds of 1:3 and $1 on $\sim P$ at (say again) odds of 1:3, then we are bound to win $2 from him, no matter what.

But this argumentation is fallacious. If somebody accepts P and views $[P] = +$, then he will not necessarily give odds on P in a bet of P vs. $\sim P$, precisely because, being a non-standard ontologist, he may *also* envisage the prospect that $[\sim P] = +$.

To accept the prospect of P and $\sim P$ conjointly is to give weight to the $\{P\} = ++$ case in a distribution of probabilities across the whole gamut of alternative cases. And so if a probability distribution is to divide a total weight of 1 across all the *standard* cases (with only non-negative values), then, over-all, we are driven to a non-standard probability theory with positive probabilities for some non-standard cases—and so with certain outcomes taking on a probability greater than 1 (and conceivably also assigning some outcomes a negative probability value—certainly so if the over-all sum is to remain at 1).

Section 5

14. See especially the powerful attack on this principle in Henry E. Kyburg, Jr., "Conjunctivists" in M. Swain (ed.), *Induction, Acceptance and Rational Belief* (Dordrecht, 1970). See also H.E. Kyburg, Jr., *Probability and the Logic of Rational Belief* (Middletown, 1961); idem "The Rule of Detachment in Inductive Logic" in I. Lakatos (ed.), *The Problem of Inductive Logic* (Amsterdam, 1968), pp. 98–119 (see especially pp. 118–119); D.M. Armstrong, *Belief, Truth and Knowledge* (Cambridge, 1973), especially chap. 13; A.H. Goldman "A Note on the Conjunctivity of Knowledge," *Analysis*, vol. 36 (1971), pp. 5–9; I. Levi, *Gambling With Truth* (New York, 1967), ch'.s II and IV; Risto Hilpinen, *Rules of Acceptance and Inductive Logic* (Amsterdam, 1968; *Acta Philosophica Fennica*, fasc. 22).

15. This is generally recognized as holding not only for acceptance-as-probably-true but (since P and Q can both be highly probable, and yet their conjunction not be so) but also for acceptance-as-approximately-true. We here extend it to acceptance-as-true proper.

16. See Alan R. Anderson and Nuel D. Belnap, Jr., *Entailment* (Princeton, 1975).

17. Various logical systems are such that the adoption of (A) engenders the unpalatable $P, \sim P \vdash Q$. (See Charles F. Kielkopf, "Adjunction and Paradoxical Derivations," *Analysis*, vol. 35 [1975], pp. 127–129.) No doubt this represents a defect in these systems. But our present approach is less radical than the rejection of (A) would be.

18. The equation of cotenability with holding in one common world goes back to Leibniz for whom (needless to say) the idea of inconsistent worlds did not arise.

19. The (T/F)-status of a molecular statement in a fusion world is determined automatically by the (T/F)-status of its components. It is not truth-conditions that are given recursively, but $[\![.]\!]$-conditions (see p. 11 above). And these latter $[\![.]\!]$-conditions in turn determine the truth-situation.

Section 7

20. The use of "*convenient*" rather than "*possible*" is deliberate here, because of the prospect of tolerating isolated inconsistencies in the interests of larger systematic considerations, an issue to be examined at some length below.

Section 8

21. Note, however, that the Tarski principle, construed in its most familiar form

P iff $T_{w*}(P)$, with $w^* =$ the actual world

will obtain only in the presence of special assumptions. The most important of these is that of the consistency of w^*, since otherwise the classical adjunction principle $P, Q \vdash P \,\&\, Q$ will at once lead *ad absurdum*. Accordingly, IF we take the view that the actual world w^* is in fact inconsistent, THEN we would be well advised to jettison the Tarski principle in its classical form.

22. These findings establish a point of kinship between the theory of inconsistent worlds and the doctrine of mathematical intuitionism. Thus it follows that with respect to such a world we may manage to demonstrate the truth of $\sim\forall x\phi x$ without being able to assure that there is any single identifiable individual a for which $\bar{\phi}$. As long as we have no *a priori* assurance of the con-

sistency of a branch of mathematics we would thus do well to abjure any temptation to resort to existence proofs by *reductio ad absurdum* within its precincts.

Section 9

23. See his essay "Ueber Gegenstandstheorie" (1904), reprinted in Vol. II of his *Gesammelte Abhandlungen* (2 vols., Leipzig, 1913–14), and tr. in R.M. Chisholm (ed.), *Realism and the Background of Phenomenology* (Glencoe, Ill., 1960). And see also his *Ueber die Stellung der Gegenstandstheorie im System der Wissenschaften* (Leipzig, 1907). For a helpful exposition of Meining's doctrine see J.N. Findlay, *Meinong's Theory of Objects and Values* (2nd ed., Oxford, 1963), especially chap. IV. Compare also the discussion of Meinong's views in R.M. Chisholm, "Beyond Being and Nonbeing" in Rudolf Haller (ed.), *Jenseits von Sein und Nichtscin* (Graz, 1972), pp. 25–36.

24. Note however that $\forall^w x \sim (\phi x \ \& \sim \phi x)$ does indeed hold unproblematically for all possible worlds.

25. For an interesting recent exchange see Karel Lambert, "Impossible Objects," *Inquiry*, vol. 17 (1974), pp. 303–314; Richard Routley, "The Durability of Impossible Objects," *ibid.*, vol. 19 (1976), pp. 247–251; Karel Lambert, "On 'The Durability of Impossible Objects,'" *ibid.*, pp. 251–253. See also: Terence Parsons, "A Prolegomenon to Meinongian Semantics," *The Journal of Philosophy*, vol. 71 (1974), pp. 561–580; Richard Routley, "Exploring Meinong's Jungle," *Notre Dame Journal of Formal Logic* (forthcoming); Richard and Valerie Routley, "Rehabilitating Meinong's Theory of Objects," *Review Internationale de Philosophie*, vol. 27 (1973), pp. 224–254; and L. Goddard and R. Routley, *The Logic of Significance and Context* (Edinburgh, 1973).

26. In Chap. 29 we are told that this letter was torn up and thrown out, in Chap. 37 that it was put away among some papers.

27. On some of the relevant historical issues see Nicholas Rescher, "The Concept of Nonexistent Possibles" in *Essays in Philosophical Analysis* (Pittsburgh, 1969), pp. 73–109 (especially 75–76). See also the interesting discussion by C.S. Peirce in *Collected Papers*, Vol. VI (Cambridge, Mass., 1935), sect. 6.19ff.

28. Contrast Gilbert Ryle's verdict "that *Gegestandstheorie* itself is dead, buried and not going to be resurrected" in Rudolf Haller, *op. cit.*, p. 7. The issuance of death-certificates is a very risky business in philosophy.

29. A reinterpretation of Meinong's theory of "objects" along the presently envisaged lines has a significant advantage over the rather different approach proposed in Terence Parsons' interesting paper, "A Prolegomenon to Meinongian Semantics," *The Journal of Philosophy*, vol. 71 (1974), pp. 561–580. For on Parsons' approach, Meinong's inconsistent objects do *not* qualify as possible existents and could not be included in the population of any possible world. However, it must be acknowledged that in *A Theory of Possibility* (Oxford, 1975), Nicholas Rescher also does not accept *schematic* individuals as genuine *individuals*, but only as abstracta corresponding to *families* of (nonschematic) individuals. (See pp. 57–58 and 97–99.)

Section 10

30. Graham Priest, "The Logic of Paradox," *The Journal of Philosophical Logic* vol. 43 (1978). It is only fair to say that there are certain solutions for certain paradoxes (e.g., the iterative conception of sets and Zermetlo's axiomatization of set theory which affects a partial formal description of this intuitive notion) which most workers of the field have ultimately come to accept as natural and intuitively plausible. But even here, the wisdom of hindsight is operative in the sense that the *independent* grounds for the solution are such that it seems unlikely that anyone would have adopted it without being driven to this by paradox. The

strength of the present approach is that while it too involves the price of an "unnatural" move (viz. rejection of the semantical adjunction principle in the presence of inconsistencies), there are independent systematic grounds for such a policy.

31. Note however that while we can have $t(Fa)$ and $t(\bar{F}a)$, we cannot have $t(F \;\&\; \bar{F}a)$. And analogously, we cannot have the truth of the *combination* of (1) and (2): $\forall x(x \in R \equiv x \notin x)$ or $\exists y \forall x(x \in y \equiv x \notin x)$. For these combination theses are refutable in first-order logic and accordingly hold in no standard worlds and thus in no schematization or superposition of such worlds. In consequence, it becomes necessary on the present approach to paradox avoidance that the Fregean Comprehension Axiom (that for any property ϕ there exists a set consisting of *all* and *only* those things which possess ϕ) must be broken apart into its two components (all, only): (i) there is a set containing all ϕ-bearers, and (ii) there is a set containing only ϕ-bearers which is a subset of any set containing all ϕ-bearers.

32. The search for changes in "naive" set theory adequate for blocking the paradoxes has thus been the motive force behind the development of most current versions of set theory. See the useful survey given in A.A. Fraenkel, Y. Bar-Hillel, and A. Levy, *Foundations of Set Theory*, 2nd ed. (Amsterdam, 1973).

33. Where we speak of "*zones* of assertion," C.S. Peirce spoke of "*fields* of assertion." (*Collected Papers*, Vol. V, sect. 5.579.) He characterized by Whewell's term "colligation" the process by which theses can be put together unproblematically to form one comprehensive conjunction:

> Those different premisses are then brought into one field of assertion, that is, are *colligated*, as Whewell [*Novum Organon Renovatum*, II, iv] would say, or conjoined into one copulative proposition. (*Loc. cit.*)

Our present thesis that that when discourse is globally inconsistent and comprehends mutually incompatible assertion zones, then this process of colligation must be confined within the boundaries of a given zone of assertion.

34. For a further elaboration of relevant ideas see Chapter V, "On the Self-Consistency of Nature" in Nicholas Rescher, *The Primacy of Practice* (Oxford, 1973).

35. It deserves note in this connection that one Soviet logician has suggested that perhaps only formal systems sufficiently rich to contain contradictions will be sufficiently rich and interesting to repay elaborate development and extensive study. See G.M. Shtshegolgova, "Paradoxien in deduktiven Systemen" in H. Wessel (ed.), *Quantoren—Modalitäten—Paradoxien* (Berlin, 1972), pp. 243–255.

36. Ample substantiation of these charges can be found in A.A. Fraenkel, Y. Bar-Hillel, and A. Levy, *Foundations of Set Theory*, 2nd ed. (Amsterdam 1973).

37. For a relatively nontechnical exposition see Ernest Nagel and James R. Newman, *Gödel's Proof* (New York, 1959).

38. The idea of a contradiction-tolerant mathematics will undoubtedly seem repugnant to many—echoing Alan Anderson's objection that if one accepts that sort of thing one is no longer doing mathematics. (See Alan Ross Anderson, "Mathematics and the 'Language Game'" in P. Benacerraf and H. Putnam [eds.], The Philosophy of Mathematics [Englewood Cliffs, 1964], see pp. 488–489.) But to take this line is to expostulate *ex cathedra* and grasp by theft a conclusion that needs to be arrived at by honest and laborious—and potentially unavailing—toil.

Section 11

39. Already Heraclitus maintained the reality of conflicts and contradictions in nature. Cf. Diels-Kranz, §22, B10, B49a, B51, etc. He taught that opposites

really do characterize the same subject. Sextus Empiricus wrote:

> Anesidemus and his followers used to say that the Sceptic Way is a road leading up to the Heraclitean philosophy, since to hold [with the Sceptics] that the same thing is the subject of opposite appearances is a preliminary to holding [with the Heracliteans] that it is the subject of opposite realities. (*Outlines of Pyrrhonism*, I, 210, tr. Bury; cf. II, 63 and compare Aristotle, *Metaphysics* 12a24ff.)

The sceptics held to the omni-indeterminacy thesis that one can never assert the truth either of *p* or of *∼p*; one can sometimes assert the truth both of p and of *∼p*. Even as the reality-is-contradiction-free school can trace its ancestry to Parmenides, so the reality-incorporates-contradictions school can claim the paternity of Heracleitus.

40. See the *Science of Logic*, II, §67. But contrast McTaggert's interpretation in *Studies in the Hegelian Dialectic* (Cambridge 1896), §8.

41. Indeed his opponents as well. See Sören Kierkegaard, *Concluding Unscientific Postscript*, tr. by D. F. Swenson (Princeton, 1941), pp. 510–11, and *Philosophical Fragments*, chap. III.

42. Cf. D.E. Berlyne, *Conflict, Arousal, and Curiosity* (New York, 1960).

43. Situations of this sort are discussed in Keith Lehrer, "Reason and Consistency" in *idem* (ed.), *Analysis and Metaphysics* (Dordrecht, 1975), pp. 57–74.

44. "A foolish consistency is the hobgoblin of little minds . . ." said Ralph Waldo Emerson in *Self-Reliance*.

45. The derivation of the paradox presupposes that 'acceptance' is acceptance *as true*, and that truths obey the standard conditions of mutual consistency, conjunctivity (i.e., that a conjunction of truths be a truth), and of closure (i.e., that the logical consequences of truths be true). The lottery paradox was originally formulated by H.K. Kyburg, Jr., *Probability and the Logic of Rational Belief* (Middletown, Conn., 1961). For an analysis of its wider implications for inductive logic see R. Hilpinen, *Rules of Acceptances and Inductive Logic* (Amsterdam, 1968; *Acta Philosophica Fennica*, fasc. 22), pp. 39–49.

46. Not, however, by Kyburg who, to his great credit, has mooted the prospect of blocking acceptance of the conjunction of an inconsistent set of accepted theses. See H.E. Kyburg, Jr., "Probability, Rationality, and a Rule of Detachment," in Y. Ban Hillel (ed.), *Proceedings of the 1964 Congress for Logic, Methodology and Philosophy of Science* (Amsterdam, 1965), pp. 203–310.

47. For suggestions regarding the utility of non-standard approaches in ethnographical studies see David E. Cooper, "Alternative Logic in 'Primitive Thought'," *Man: Journal of the Royal Anthropological Institute*, vol. 10 (1958), pp. 238–256.

48. For a rather different approach see N. Rescher, *The Coherence Theory of Truth* (Oxford, 1973).

49. D.C. Makinson, "The Paradox of the Preface," *Analysis*, 25 (1964), 205–7. Compare H.E. Kyburg, Jr., "Conjunctivitis" in M. Swain (ed.), *Induction, Acceptance, and Rational Belief* (Dordrecht, 1970), pp. 55–82, see esp. p. 77; and also R.M. Chisholm, *The Theory of Knowledge*, 2nd ed. (Englewood Cliffs, 1976), pp. 96–97. The fundamental idea of the Preface Paradox goes back to C.S. Peirce, who wrote: "that while holding certain propositions to be each individually perfectly certain, we may and ought to think it likely that some of them, if not more, are false." (*Collected Papers*, 5.498.)

50. Keith Lehrer, *Knowledge* (Oxford, 1974), p. 203.

51. An amusing but vividly clear picture of the problem is given in John G. Saxe's poem "The Blind Men and the Elephant" which tells the story of the wise men of Indostan who investigated the elephant:

. . . six men of Indostan,
To learning much inclined,
Who went to see the elephant,
(Though all of them were blind).

One sage stumbled against the elephant's "broad and sturdy side" and declared the beast to be "very like a wall." Another, who had felt its tusk, pronounced the elephant to be very like a spear. The third, who took the elephant's squirming trunk in his hands, compared it to a snake; while the fourth, who put his arms around the elephant's knee, was sure that the animal resembled a tree. A flapping ear convinced another that the elephant had the form of a fan; while the sixth blind man was convinced that it had the form of a rope, since he took hold of the tail.

And so these men of Indostan,
Disputed loud and long;
Each in his own opinion
Exceeding stiff and strong:
Though each was partly in the right,
And all were in the wrong.

52. Just here lies the profound lesson of the story of the blind men and the elephant. The inconsistencies at issue do not result from "the data" available to the men—what they feel and experience. It is their systematizing extensipn of these data that produces the conflict.

53. Eugene P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," *Communications on Pure and Applied Mathematics*, vol. 13 (1960), pp. 1–14 (see pp. 11–12). Wigner has suggested in private conversation that a more radical disunity is at issue. The space-time metric of general relativity requires mathematically *punctiform* occurrence-configurations, whereas the quantum theory excludes the prospect of such point-events. The requirements of the two domains are to all appearances incompatible with one another.

54. The situation is reminiscent of the late 19th-century split between physicists (especially William Thompson, later Lord Kelvin) on the one hand and geologists and biologists (especially T.H. Huxley on the other) over the issue of the age of the earth. See the discussion in Stephen G. Brush, "Science and Culture in the Nineteenth Century," *The Graduate Journal*, vol. 7 (1969), pp. 479–565.

55. That is, considerations of systemic loss and gain can so operate that the costs of consistency-preservation are such that "it's not worth it."

56. The discussion of this issue is indebted to Keith Lehrer, "Reason and Consistency" in *idem* (ed.), *Analysis and Metaphysics* (Dordrecht, 1975), pp. 57–74.

57. If inconsistencies can be localizable singularities, as we have argued, these may, of course, be more or less numerous.

58. Cf. Arthur Fine, "Some Conceptual Problems of Quantum Mechanics" in R. Colodny (ed.), *Paradigms and Paradoxes* (Pittsburgh, 1972).

Section 12

59. Moreover, were one to follow C.I. Lewis in viewing consistency as a species of *uniformity* (in that what is to be said and done in all inherently uniform cases remains invariant), then the prospect of encountering inconsistencies within the orbit of the real becomes more readily acceptable. See C.I. Lewis, *Values and Imperatives*, ed. by J. Lange (Stanford, 1969).

60. Compare Hegel's observation that "In Scepticism consciousness gets, in truth, to know itself as a consciousness containing contradition within itself." (*Phenomenology of Mind*, tr. G. Lichtheim [New York, 1967], p. 250.)

61. See, for example, *Anti-Dühring*, tr. by E. Burns (New York, 1939), p. 132.

62. Nicholas Lobkowicz, "The Principle of Contradiction in Recent Soviet

Philosophy," *Studies in Soviet Thought*, vol. 1 (1961), pp. 44–50.

63. For a full account see Jürg Hänggi, *Formale und dialektische Logik in der Sowjetphilosophie* (Winterthur, 1971). And a comprehensive bibliography is given in this same author's *Bibliographie der Sowjetischen Logik* (Winterthur, 1971). For critical surveys of relevant literature see J.M. Bochenski, "Soviet Logic," *Studies in Soviet Thought*, vol. 1 (1961), pp. 29–38; T.J. Blakeley, review article, *Studies in Soviet Thought*, vol. 3 (1963), pp. 165–166; Nicholas Lobkowicz, *Das Wiederspruchsprinzip in der neueren sowjetischen Philosophie* (Dordrecht, 1959) and "The Principle of Contradiction in Recent Soviet Philosophy," *Studies in Soviet Thought*, vol. 1 (1961), pp. 44–50; G.A. Wetter, *Soviet Ideology Today* (New York, 1966); David Dinsmore Comey, "Current Trends in Soviet Logic," *Inquiry*, vol. 9 (1966), pp. 94–108; J.M. Bochenski, *The Dogmatic Principles of Soviet Philosophy (as of 1958)* (Dordrecht, 1963); G.L. Kline's review article (of articles in *Voprosy filosofii* discussing the relationship of formal logic and dialectical logic), *The Journal of Symbolic Logic*, vol. 18 (1958), pp. 83–86; H.B. Acton, "Dialectical Materialism" in *The Encyclopedia of Philosophy*, ed. by Paul Edwards, Vol. 2 (New York, 1967), pp. 384–397.

64. For example, Alonzo Church unblushingly cites "the incompatibility of logic and dialectic" as a principal reason why "Soviet logic will not develop fruitfully or successfully." (Review of A. Philipov's *Logic and Dialectic in the Soviet Union, Journal of Symbolic Logic*, vol. 18 (1958), pp. 272–273.)

65. See, for example, Michael Kosok, "The Formalization of Hegel's Dialectical Logic," *International Philosophical Quarterly*, vol. 6 (1966), pp. 596–631; reprinted in A. MacIntyre (ed.), *Hegel: A Collection of Critical Essays* (Garden City, 1972). See also Dominique Dubarle and André Doz, *Logique et dialectique* (Paris, 1972), and A. Sarlemijn, *Hegel's Dialectic* (Dordrecht, 1975), which offer helpful bibliographies.

66. A useful survey is given in Robert G. Wolf, *Contradictions and Logical Systems* (unpublished).

67. Stanislaw Jaskowski, "Un calcul des propositions pour les systèmes deductifs contradictoires," *Studia Societatis Scientiarum Torunensis*, vol. 1 (1948), pp. 55–57.

68. The starting point was da Costa's 1963 doctoral dissertation at the Federal University of Parana: *Sistemas formais inconsistentes*. A useful recent exposition of the work of da Costa's and his collaborators is given in his paper, "On the Theory of Inconsistent Formal Systems," *Notre Dame Journal of Formal Logic*, vol. 15 (1974), pp. 497–510, which gives a full bibliography.

69. F.G. Asenjo, "A Calculus of Antinomies," *Notre Dame Journal of Formal Logic*, vol. 7 (1966), pp. 103–105.

70. S.K. Thomason, "Towards a Formalization of Dialectical Logic," forthcoming. (See the abstract in the *Journal of Symbolic Logic*, vol. 39 [1974], p. 204.)

71. Richard and Valerie Routley, *Beyond the Actual* (forthcoming). Richard Routley and Robert K. Meyer, "Dialectical Logic, Classical Logic, and the Consistency of the World," *Dialectics and Humanism* (forthcoming).

72. Meyer's joint paper with Routley and Meyer see Richard and Valerie Routley (*op. cit.*), which gives a forceful and cogent defense of the aims and prospects of dialectical logic.

73. "The Logic of Paradox," *Journal of Philosophical Logic* (forthcoming).

74. See Alan R. Anderson and Nuel D. Belnap, Jr., *Entailment*, vol. I (Princeton, 1975).

75. For an excellent semi-popular account of the Everett-Wheeler theory see B.S. DeWitt, "Quantum Mechanics and Reality," *Physics Today* (Sept., 1970), pp. 30–35.

76. Compare the alternative treatment proposed in Nicholas Rescher, *The Primacy of Practice* (Oxford, 1973), a treatment based on a differentiation with respect to the time-dimension.

77. The recent literature includes: Hector Neri Castaneda, "Thinking and the Structure of the World," unpublished; Roderick M. Chisholm, "Beyond Being and Nonbeing" in Rudolf Haller (ed.), *Jenseits von Sein und Nichtsein: Beiträge zur Meinong-Forschung* (Graz, 1972), pp. 25–36; *idem*, "Homeless Objects," *Revue internationale de Philosophie*, vol. 104/5 (1973), pp. 207–223; J.N. Findlay, *Meinong's Theory of Objects and Values*, 2nd ed. (Oxford, 1963); L. Goddard and R. Routley, *The Logic of Significance and Context*, vol. 1 (Aberdeen, 1973); Reinhardt Grossman, *Meinong* (London, 1974), Guide Küng, "Noema und Gegenstand" in Rudolf Haller (ed.), *Jenseits von Sein und Nichtsein: Beiträge zur Meinong-Forschung* (Graz, 1972), pp. 55–62; Karel Lambert, "Impossible Objects," *Inquiry*, vol. 17 (1974), pp. 303–314; *idem*, "On 'The Durability of Impossible Objects'," *Inquiry*, vol. 19 (1976), pp. 251–253; *idem*, "Review Discussion: The Theory of Objects," *Inquiry*, vol. 16 (1973), pp. 221–230; Terence Parsons, "A Prolegomenon to Meinongian Semantics," *The Journal of Philosophy*, vol. 71 (1974), pp. 561–580; Richard Routley, "Exploring Meinong's Jungle: Items and Descriptions," *Notre Dame Journal of Formal Logic*, forthcoming; *idem*, "The Durability of Impossible Objects," *Inquiry*, vol. 19 (1976), pp. 247–251; Richard and Valerie Routley, "Rehabilitating Meinong's Theory of Objects," *Revue Internationale de Philosophie*, vol. 27 (1973), pp. 224–254; S.K. Thomason, "Towards a Formulation of Dialectical Logic" (forthcoming).

Section 13

78. Saul Kripke, "Semantical Considerations on Modal Logics," *Acta Philosophica Fennica*, vol. 16 (1963).

79. Carnap, Meaning and Necessity: *A Study in Semantics and Modal Logic* (Chicago, 1947).

80. Lewis and Langford, *Symbolic Logic* (New York, 1932).

81. In the dispute over the existence of possibilia, a similar strategy emerged. For in spite of Kripke's demonstration of the value of a meta-theory for modal discourse which quantifies over possible worlds, the suspicion remained that that meta-theory could be formulated in such a way as to avoid such quantificational commitment. This thesis is clearly most plausible in those circumstances in which all the modalities our meta-theory seeks to explain are modalities *de dicto*. Attention accordingly turned to the question of *de re* modalities, that is, to the question to what extent it is necessary or useful to allow quantification over possible individuals in our modal discourse. Insofar as a form of discourse is used that can only be interpreted as involving quantification over possible individuals, one is commited not only to the existence of such individuals, but obviously to the existence of possible worlds which they inhabit or comprise as well. We will have nothing to say regarding the dispute about *possible* individuals *per se*, but will show that it is coherent and potentially useful to quantify over inconsistent and incomplete individuals.

Section 17

82. "Naming and Necessity" in G. Harmon and D. Davidson (eds.) *Semantics of Natural Language* (Dordrecht, 1972), pp. 266–267.

83. Compare N. Rescher, *A Theory of Possibility* (Oxford, 1975).

Section 18

84. G. Birkhoff, *Lattice Theory* (American Mathematical Society, 1940).

85. See Abraham Robinson, *Non-Standard Analysis* (Amsterdam, 1965).

86. Both these posets are in fact lattices as well, since meets and joins satisfying Ll–L4 can be defined on them in a natural way.

Section 20

87. *Word and Object* (Cambridge, Mass., 1960), p. 23.

Section 21

88. Cf., Peirce, *Collected Papers*, Sect. 5.378 ("The Fixation of Belief").

Section 22

89. In establishing a consensus methodology, the mechanism of inconsistent and schematic worlds can be of considerable utility. Suppose for example we consider the views of the equicredible sources with respect to the two propositions *p* and *q*, with the following result

	Source 1	*Source 2*
p	true	false
q	true	true

We then may arrive either at the inconsistent "consensus world" in which *q* is true but *p* is *both* true or false (if we accept the declarations of both sources "at face value") or else we will arrive at the schematic "consensus world" in which *q* is true but *p* indeterminate (if we take the conflict with respect to *p* as cancelling out the information regarding *p* provided by the sources.

90. One final note concerning the various complex constructions out of double convergence which might be envisioned to represent the respects and relations of communal agreement which matter for cognitive inquiries: Although out official definition of convergence employs *nets* of possible worlds, until now only convergent *sequences* (a very special case of convergent nets) have been considered. Even double-convergence is defined in terms of the mutual convergence of two sequences. It can instead be considered as the single convergence of a net. More complicated constructions using the principle of double convergence to represent more complex communal inquiries will undoubtedly use the more general notion, since not all the claims statable in terms of nets are paraphrasable into statements about sequences. But this is a formal resource of our construction which we will not exploit here.

Section 23

91. Cf. Peirce's views on this theme, e.g., at *Collected Papers*, 5.384.

92. See Hilary Putnam, "The Refutation of Conventionalism," in his *Mind, Language, and Reality: Philosophical Papers Volume II* (Cambridge, 1976).

93. *Science and Metaphysics* (London, 1968), Chapter 5. See also J. Rosenberg, *Linguistic Representation* (Dordrecht, 1974), Chapter 5.

94. *Against Method* (London, 1976).

Section 24

95. Cf. *Collected Papers*, 8.15.

96. This remark should be compared with Kant's assertion of the empirical reality and transcendental ideality of space and time—though of course for Peirce the constitution of that methodological reality is by means of a *social* inquiry controlled by a *public* methodology.

97. Cf. *Collected Papers*, 5.429.

98. *Collected Papers*, 8.12–8.17.

99. *Collected Papers*, 5.316–5.317.

Section 25

100. Note that the parity thesis of the present discussion merely maintains that the "ontological status" of standard and nonstandard worlds is to be the same, without involving any commitment on the issue of just what this status is. For a discussion of this issue see N. Rescher, *A Theory of Possibility* (Oxford, 1973).

101. See David Lewis, "Anselm and Actuality," *Nous*, vol. 4 (1970), pp. 175–188.

Section 26

102. Recall F.H. Bradley's dictum: "If by taking certain judgments . . . as true I can get more system into my world, then these facts are so far true. . . ." (*Essays on Truth and Reality* [Oxford, 1914], p. 210.)

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